J. Fluid Mech. (1998), vol. 361, pp. 41–74. Printed in the United Kingdom

Hydrodynamic equations for rapid flows of smooth inelastic spheres, to Burnett order

By N. SELA and I. GOLDHIRSCH
Department of Fluid Mechanics and Heat Transfer, Faculty of Engineering, Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel

(Received 14 February 1997 and in revised form 18 November 1997)

The Chapman–Enskog expansion is generalized in order to derive constitutive relations for flows of inelastically colliding spheres in three dimensions – to Burnett order. To this end, the pertinent (nonlinear) Boltzmann equation is perturbatively solved by performing a (double) expansion in the Knudsen number and the degree of inelasticity. One of the results is that the normal stress differences and the 'temperature anisotropy', characterizing granular fluids, are Burnett effects. The constitutive relations derived in this work differ, both qualitatively and quantitatively, from those obtained in previous studies. In particular, the Navier–Stokes (order) terms have a different dependence on the degree of inelasticity and the number density than in previously derived constitutive relations; for instance, the expression for the heat flux contains a term which is proportional to $\epsilon\nabla \log n$, where ϵ is a measure of the degree of inelasticity and n denotes the number density. This contribution to the heat flux is of zeroth order in the density; a similar term, i.e. one that is proportional to $\epsilon\nabla n$, has been previously obtained by using the Enskog correction but this term is $O(n)$ and it vanishes in the Boltzmann limit. These discrepancies are resolved by an analysis of the Chapman–Enskog and Grad expansions, pertaining to granular flows, which reveals that the quasi-microscopic rate of decay of the temperature, which has not been taken into account heretofore, provides an important scale that affects the constitutive relations. Some (minor) quantitative differences between our results and previous ones exist as well. These are due to the fact that we take into account an isotropic correction to the leading Maxwellian distribution, which has not been considered before, and also because we consider the full dependence of the corrections to the Maxwellian distribution on the (fluctuating) speed.

1. Introduction

Owing to the recent significant increase in interest in granular systems in the scientific community, a paper in this field no longer needs to explain the importance of granular materials and their wide range of applicability. Part of the recent work in this field (see Herrmann 1992; Hutter & Rajagopal 1994; Jaeger, Nagel & Behringer 1996 and references therein for recent reviews) deals with the quasi-static flow regime, which is usually characterized by relatively large densities, relatively prolonged contacts among the grains and more than two-body interactions. The 'opposite' limit, in which the particle interactions are binary collisions (characterized by short durations), is coined 'rapid granular flow' (cf. the review by Campbell 1990 and references therein). It is known that a given system may contain domains in which the flow is quasi-static and regions in which it is 'rapid' (Campbell 1990; Herrmann 1992; Hutter & Rajagopal 1994; Jaeger *et al.* 1996). Equations derived for rapid granular flows may be of some use in the quasi-static regime since one of the major factors that needs to be taken into account is the dissipative nature of the interparticle interactions, a feature which is common to all regimes of granular flows.

The present paper deals with rapid granular flows. As noticed in numerous previous studies, e.g. in Lun *et al.* (1984); Jenkins & Richman (1985*a,b*, 1988); Haff (1986); Campbell (1990); Lun (1991), the similarity between granular materials in a state of rapid flow and the classical picture of molecular gases suggests that methods used in the realm of statistical mechanics of gases may be relevant to the analysis of rapid granular flows. The Boltzmann equation and its (Enskog) generalization to (moderately) dense flows have been employed (Lun *et al.* 1984; Jenkins & Richman 1985*a,b*, 1988; Boyle & Massoudi 1990 and Lun 1991) in order to obtain constitutive relations for granular systems. Most of the kinetic studies do not involve a direct and/or systematic analysis of the pertinent Boltzmann equation (see however Goldshtein & Shapiro 1995 for a direct analysis). Instead, an ansatz for the form of the single-particle distribution function has been substituted in the (Enskog) equations of motion obeyed by the low moments (corresponding to the mass density, momentum density and kinetic energy density) of the distribution function. The Grad method of moments (Grad 1949), which involves a closure to render it useful, has also been employed in this field (Jenkins & Richman 1985*a*).

One of the aims of the present paper to show how a systematic perturbative solution of the Boltzmann equation for inelastically colliding particles can be obtained; more specifically, we study the case of a monodisperse collection of smooth spheres interacting by binary collisions, characterized by a fixed coefficient of normal restitution. The expansion of the single-particle distribution is used to derive constitutive relations for the above system and draw some additional conclusions.

The method employed below is a generalization of the Chapman–Enskog expansion (Kogan 1969; Chapman & Cowling 1970; Harris 1971; Cercignani 1975; Goldhirsch & Sela 1996 and Sela, Goldhirsch & Noskowicz 1996). The need for such a generalization has been explained in previous publications (Goldhirsch & Sela 1996 and Sela *et al.* 1996) and we shall only briefly repeat the explanation. The standard Chapman–Enskog expansion for molecular systems employs the (time-independent, exact) equilibrium solution of the Boltzmann equation as its zeroth order; shearing, temperature gradients and the like are considered as perturbations and the zeroth-order solution corresponds to an unforced, or free, gas. Such a zeroth-order solution does not exist for granular flows since in a free granular system the kinetic energy decays, due to the inelasticity of the collisions, to an asymptotic state of zero granular temperature – the only unforced steady-state of such a flow. The latter state is clearly an inadequate candidate for a 'zeroth-order solution' when one is interested in a forced system at finite granular temperature. In previous studies (Goldhirsch & Sela 1996 and Sela *et al.* 1996) we have shown how a perturbative approach to solving the Boltzmann equation can be designed, in spite of the above problem. The idea underlying the resolution of this problem is conveniently demonstrated in the case of a simple steady shear flow. In this case the increase in the granular temperature (heating) by shear is compensated by the energy losses (cooling) due to the inelasticity of the collisions. The equality of the rates of heating by shear and cooling by inelasticity (in a steady state) can be shown (Haff 1986; Goldhirsch & Sela 1996 and Sela *et al.* 1996) to yield: $T \propto \gamma^2\ell^2/\epsilon$ where T is the granular temperature, ℓ is the mean free path, γ is the shear rate and ϵ (which equals $1-e^2$, where e is the coefficient of normal restitution) is a measure of the degree of inelasticity. One observes that the double limit $\epsilon \to 0$ and $\gamma \to 0$, in which one keeps the ratio γ^2/ϵ fixed, corresponds to an equilibrium state (whose temperature is proportional to the ratio γ^2/ϵ). This limit is not singular: the energy loss in a given collision is proportional to ϵ and (local) equilibration occurs on the time scale of a few collisions (mean free times). This observation has served as the basis of a perturbative expansion (Sela *et al.* 1996) of the Boltzmann equation for a simply sheared granular system in powers of $\epsilon^{1/2}$ around the equilibrium solution (as a 'zeroth-order' solution). This approach has been further generalized for the case of a time-dependent homogeneous (two-dimensional) granular system (Goldhirsch & Sela 1996). Since a granular system can 'reach' an equilibrium state when *both* γ and ϵ vanish, it is possible to define a double perturbative expansion in which *both* γ (properly non-dimensionalized, see below) and ϵ serve as small parameters. The latter expansion reduces to that corresponding to the above steady-state solution in the appropriate limit. One of the major results obtained from Goldhirsch & Sela (1996) is a qualitative and quantitative understanding of the normal stress differences in granular systems; it turns out that this property of granular systems is a Burnett effect and it corresponds to a similar property of molecular systems, the only difference being of quantitative nature: the Burnett effect in granular systems is about 20 orders of magnitude larger than in simple molecular systems and it is observable (Goldhirsch & Sela 1996).

The study presented below is a further generalization of the above investigations; an expansion in which the Knudsen number and the degree of inelasticity are the small parameters is performed for a three-dimensional system of smooth spheres and carried out to Burnett order. This expansion yields several results which are different from what has been obtained heretofore: (i) The heat flux contains a term of order ϵ which is proportional to $\nabla \log n$, where n is the number density; a similar term appears in previous works (Jenkins & Richman 1985*a*; Boyle & Massoudi 1990 and Lun 1991) either at higher orders in ϵ or as a result of using the Enskog correction (beyond the Boltzmann level of description) and then it is proportional to ∇n and not $\nabla \log n$. (ii) An $O(\epsilon)$ term is obtained in the expression for the coefficient of viscosity whereas in the previous work mentioned above the lowest-order ϵ-dependent term in the expression for the viscosity is $O(\epsilon^2)$. (iii) The $O(\epsilon)$ term in the expression for the heat conductivity is positive whereas previous work produced a negative term. In addition, some prefactors are different as well. The reason for the discrepancy between the constitutive relations obtained in the present work and those obtained in previous studies is shown below to be related to the fact the dynamics of the granular temperature field defines a relevant quasi-microscopic (or short) time scale. A careful analysis of the equations derived in previous studies, in particular by employing Grad's method of moments, reveals that when the correct dynamics of the temperature is taken into account, one obtains results which are in agreement with our findings; some remaining minor quantitative differences are explained below. We wish to stress that some of the pioneering studies of rapid granular flows may not have been intended to be accurate to $O(\epsilon)$; these important works led the way to the rational approach presented here.

The structure of the paper is as follows: §2 presents the system to be studied along with the pertinent Boltzmann equation and some general results. The perturbative expansion is set up in §3. The resulting constitutive relations, to Burnett order, are presented in §3.5, where the normal stress difference is shown to be a Burnett effect. Section 4 provides an explanation for the discrepancies between our results and previous ones. Concluding remarks, as well as comments concerning necessary future work, are presented in §5. Many of the technical details are relegated to a set of Appendices. Appendices A, D and E are not printed here but are available from the authors or the JFM Editorial Office.

2. Formulation of the problem

The present article deals with monodisperse collections of smooth inelastically colliding spheres of diameter d, whose collisions are characterized by a *constant* coefficient of normal restitution, e, which satisfies $0 < e \leqslant 1$. The binary collision between spheres labelled i and j results in the following velocity transformation:

$$\boldsymbol{v}_i = \boldsymbol{v}'_i - \frac{1+e}{2}(\hat{\boldsymbol{k}} \cdot \boldsymbol{v}'_{ij})\hat{\boldsymbol{k}}, \tag{1}$$

where $(\boldsymbol{v}'_i, \boldsymbol{v}'_j)$ are the precollisional velocities, $(\boldsymbol{v}_i, \boldsymbol{v}_j)$ are the corresponding postcollisional velocities, $\boldsymbol{v}'_{ij} \equiv \boldsymbol{v}'_i - \boldsymbol{v}'_j$ and $\hat{\mathbf{k}}$ is a unit vector pointing from the centre of sphere i to that of sphere j at the moment of contact.

The properties of the system (assuming it is dilute enough) are assumed to be described by the Boltzmann equation (Grad 1949; Kogan 1969; Chapman & Cowling 1970; Harris 1971; Cercignani 1975; Goldshtein & Shapiro 1995; Sela *et al.* 1996 and Goldhirsch & Sela 1996)

$$\frac{\partial f}{\partial t} + \boldsymbol{v}_1 \cdot \nabla f = d^2 \int_{\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12} > 0} d\boldsymbol{v}_2 \mathrm{d}\hat{\boldsymbol{k}} (\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12}) \left(\frac{1}{e^2} f(\boldsymbol{v}'_1) f(\boldsymbol{v}'_2) - f(\boldsymbol{v}_1) f(\boldsymbol{v}_2) \right), \tag{2}$$

where $f \equiv f(\boldsymbol{v}_1, \boldsymbol{r}, t)$ is the single-particle distribution function, ∇ is a gradient with respect to the spatial coordinate $\boldsymbol{r}$ and the other variables are defined in (1) and the text following it. The dependence of f on the spatial coordinates and on time is not explicitly spelled out in (2). Notice that in addition to the explicit dependence of (2) on e, it also implicitly depends on e through the relation between the postcollisional and precollisional velocities.

The hydrodynamic variables considered below are (Grad 1949; Kogan 1969; Chapman & Cowling 1970; Harris 1971 and Cercignani 1975): the number density field, $n(\boldsymbol{r}, t)$, the macroscopic velocity field, $\boldsymbol{V}(\boldsymbol{r}, t)$, and the granular temperature field, $\Theta(\boldsymbol{r}, t)$. These quantities are given by:

$$n(\boldsymbol{r}, t) \equiv \int \mathrm{d}\boldsymbol{v} f(\boldsymbol{v}, \boldsymbol{r}, t), \tag{3}$$

$$\boldsymbol{V}(\boldsymbol{r}, t) \equiv \frac{1}{n} \int \mathrm{d}\boldsymbol{v} \boldsymbol{v} f(\boldsymbol{v}, \boldsymbol{r}, t), \tag{4}$$

and

$$\Theta(\boldsymbol{r}, t) \equiv \frac{1}{n} \int \mathrm{d}\boldsymbol{v} (\boldsymbol{v} - \boldsymbol{V})^2 f(\boldsymbol{v}, \boldsymbol{r}, t). \tag{5}$$

respectively; also $1/n$ denotes $1/n(\boldsymbol{r}, t)$. In this article the mass, m, of a particle, is normalized to unity. The granular temperature, defined in (5) (without the factor $\frac{1}{3}$ often used in the literature), is a measure of the squared fluctuating velocity. This definition does not contradict the common notion of a tensorial granular temperature (see Jenkins & Richman 1988 and Campbell 1990) which is a way of expressing the fact that the stress tensor in sheared flows is anisotropic. This anisotropy is one of the results of our analysis and one does not have to use it as input (see below). The equations of motion for the above-defined macroscopic field variables can be formally derived by multiplying the Boltzmann equation, (2), by 1, $\boldsymbol{v}_1$ and v_1^2 respectively, and integrating over $\boldsymbol{v}_1$. A standard procedure (which employs the symmetry properties of the collision integral on the right-hand side of the Boltzmann equation) yields equations of motion for the hydrodynamic fields (Lun *et al.* 1984; Jenkins & Richman 1985*a,b*, 1988; Boyle & Massoudi 1990 and Lun 1991):

$$\frac{\mathrm{D}n}{\mathrm{D}t} + n\frac{\partial V_i}{\partial r_i} = 0, \tag{6}$$

$$n\frac{\mathrm{D}V_i}{\mathrm{D}t} + \frac{\partial P_{ij}}{\partial r_j} = 0, \tag{7}$$

$$n\frac{\mathrm{D}\Theta}{\mathrm{D}t} + 2\frac{\partial V_i}{\partial r_j}P_{ij} + 2\frac{\partial Q_j}{\partial r_j} = -n\Gamma, \tag{8}$$

where $\boldsymbol{u} \equiv \boldsymbol{v} - \boldsymbol{V}$ is the fluctuating velocity, $P_{ij} \equiv n\langle u_i u_j \rangle$ is the stress tensor, $Q_j \equiv \frac{1}{2}n\langle u^2 u_j \rangle$ is the heat flux vector, $\langle\ \rangle$ is an average with respect to f, $\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + \boldsymbol{V}\cdot\nabla$ is the material derivative and Γ, which accounts for the energy loss in the (inelastic) collisions, is given by

$$\Gamma \equiv \frac{\pi(1-e^2)d^2}{8n}\int \mathrm{d}\boldsymbol{v}_1 \mathrm{d}\boldsymbol{v}_2 v_{12}^3 f(\boldsymbol{v}_1) f(\boldsymbol{v}_2). \tag{9}$$

Notice that (6)–(8) are *exact* consequences of the Boltzmann equation. The microscopic details of the interparticle interactions affect the values of the averages $\langle u_i u_j \rangle$, $\langle u^2 u_i \rangle$ and Γ. A standard method for obtaining these quantities for molecular gases is the Chapman–Enskog expansion (Kogan 1969; Chapman & Cowling 1970; Harris 1971 and Cercignani 1975). It involves a perturbative solution of the Boltzmann equation in powers of the spatial gradients of the hydrodynamic fields; the zeroth-order solution yields the Euler equations, the first order gives rise to the Navier–Stokes equations, the second order begets the Burnett equations, etc. The Chapman–Enskog method is tailored for systems that have a stationary homogeneous (equilibrium) solution, which serves as a zeroth-order solution of the expansion. Since granular systems do not possess such equilibrium-like solutions (an unforced homogeneous system is one in which energy decays due to the inelasticity of the collisions) the Chapman–Enskog technique is not directly applicable to such systems; as shown below, an appropriate generalization of the Chapman–Enskog expansion can be employed in the realm of granular systems.

3. Method of solution

The method of solution employed in this work is a generalization of the Chapman–Enskog expansion. The classical Chapman–Enskog expansion assumes the smallness of the Knudsen number, $K \equiv \ell/L$ where ℓ is the mean free path given by $\ell = 1/(\pi n d^2)$ and L is a macroscopic length scale, i.e. the length scale which is resolved by hydrodynamics, not necessarily the system size. Here we define a second small parameter, ϵ, given by $\epsilon \equiv 1 - e^2$, which is a measure of the inelasticity and we (formally) assume here: $\epsilon \ll 1$. Next, we perform a rescaling of the Boltzmann equation, as follows: spatial gradients are rescaled as $\nabla \equiv (1/L)\tilde{\nabla}$, the rescaled fluctuating velocity is $\tilde{\boldsymbol{u}} \equiv (3/(2\Theta))^{1/2}(\boldsymbol{v} - \boldsymbol{V})$ and $f \equiv n\left(3/(2\Theta)\right)^{3/2}\tilde{f}(\tilde{\boldsymbol{u}})$. In terms of the rescaled quantities, the Boltzmann equation assumes the form

$$\tilde{\mathscr{D}}\tilde{f} + \tilde{f}\tilde{\mathscr{D}}\left(\log n - \tfrac{3}{2}\log\Theta\right) = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2') - \tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\right) \equiv \tilde{\mathscr{B}}(\tilde{f},\tilde{f},e), \quad (10)$$

where

$$\tilde{\mathscr{D}} \equiv K\left(\frac{3}{2\Theta}\right)^{1/2}\left(L\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\tilde{\nabla}\right). \quad (11)$$

Notice that $\tilde{\mathscr{D}}$ is not a material derivative since the velocity $\boldsymbol{v}$ is not the hydrodynamic velocity but rather the particle's velocity. Clearly, the double limit $\epsilon \to 0$ and $K \to 0$, with constant number density, corresponds to a homogeneous elastically colliding collection of spheres for which the distribution function is Maxwellian. Hence, for $K \ll 1$ and $\epsilon \ll 1$, $\tilde{f}$ can be expressed as follows: $\tilde{f}(\tilde{\boldsymbol{u}}) = \tilde{f}_0(\tilde{u})(1+\Phi)$ where $\tilde{f}_0(\tilde{u}) = \pi^{-3/2}\mathrm{e}^{-\tilde{u}^2}$ and Φ is considered to be a 'small' perturbation. Employing the above form of $\tilde{f}$ and making use of $\tilde{u}^2 = 3(\boldsymbol{v}-\boldsymbol{V})^2/2\Theta$ it follows that (10) can be transformed to

$$(1+\Phi)\left(\tilde{\mathscr{D}}\log n + 2\left(\frac{3}{2\Theta}\right)^{1/2}\tilde{u}_i\tilde{\mathscr{D}}V_i + \left(\tilde{u}^2 - \tfrac{3}{2}\right)\tilde{\mathscr{D}}\log\Theta\right) + \tilde{\mathscr{D}}\Phi = \frac{1}{\tilde{f}_0}\tilde{\mathscr{B}}(\tilde{f},\tilde{f},e). \quad (12)$$

The following relations follow directly from (6)–(8) and the definition of $\tilde{\mathscr{D}}$:

$$\tilde{\mathscr{D}}\log n = K\left(\tilde{u}_i\frac{\partial\log n}{\partial\tilde{r}_i} - \left(\frac{3}{2\Theta}\right)^{1/2}\frac{\partial V_i}{\partial\tilde{r}_i}\right), \quad (13)$$

$$\tilde{\mathscr{D}}V_i = K\left(\tilde{u}_j\frac{\partial V_i}{\partial\tilde{r}_j} - \frac{1}{n}\left(\frac{3}{2\Theta}\right)\frac{\partial P_{ij}}{\partial\tilde{r}_j}\right), \quad (14)$$

and

$$\tilde{\mathscr{D}}\log\Theta = K\left(\tilde{u}_j\frac{\partial\log\Theta}{\partial\tilde{r}_j} - \frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{1/2}P_{ij}\frac{\partial V_i}{\partial\tilde{r}_j} - \frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{1/2}\frac{\partial Q_j}{\partial\tilde{r}_j}\right) - \epsilon\tilde{\Gamma}, \quad (15)$$

where

$$\tilde{\Gamma} \equiv \frac{1}{12}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\tilde{u}_{12}^3\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2). \quad (16)$$

Next we expand Φ in both small parameters, ϵ and K: $\Phi = \Phi_K + \Phi_\epsilon + \Phi_{KK} + \Phi_{K\epsilon} + \ldots$ where here, and in the rest of the paper, subscripts indicate the order of the corresponding terms in the small parameters, e.g. $\Phi_K = O(K)$. It is perhaps worthwhile mentioning that the $O(K\epsilon^n)$, for all $n \geqslant 0$, corrections to the single-particle distribution function are named the Navier–Stokes or Chapman–Enskog order whereas the $O(K^2\epsilon^n)$ corrections are Burnett terms. In parallel to the expansion of Φ in the small parameters, the operation of $\tilde{\mathscr{D}}$ on any function of the field variables, ψ, can be formally expanded as the following sum: $\tilde{\mathscr{D}}\psi = \tilde{\mathscr{D}}_K\psi + \tilde{\mathscr{D}}_\epsilon\psi + \tilde{\mathscr{D}}_{KK}\psi + \tilde{\mathscr{D}}_{K\epsilon}\psi + \tilde{\mathscr{D}}_{\epsilon\epsilon}\psi + \ldots$, where e.g. $\tilde{\mathscr{D}}_{K\epsilon}\psi$ is the $O(K\epsilon)$ term in the expansion of $\tilde{\mathscr{D}}\psi$ in powers of K and ϵ. Since this expansion is well defined we shall refer to the symbols $\tilde{\mathscr{D}}_K$, $\tilde{\mathscr{D}}_\epsilon$ etc. as operators in their own right.

3.1. *Solution at $O(K)$*

Upon substituting $e = 1$ (or $\epsilon = 0$) in the right-hand side of (12) and retaining only $O(K)$ terms, one obtains

$$\tilde{\mathscr{L}}(\Phi_K) = \tilde{\mathscr{D}}_K \log n + 2\left(\frac{3}{2\Theta}\right)^{1/2} \tilde{u}_i \tilde{\mathscr{D}}_K V_i + \left(\tilde{u}^2 - \tfrac{3}{2}\right) \tilde{\mathscr{D}}_K \log\Theta \tag{17}$$

where $\tilde{\mathscr{L}}$ is the (standard) rescaled linearized Boltzmann operator (Kogan 1969; Chapman & Cowling 1970; Harris 1971 and Cercignani 1975) for elastically colliding particles, given by

$$\tilde{\mathscr{L}}(\Phi) \equiv \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\mathrm{d}\tilde{\boldsymbol{u}}_2(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\mathrm{e}^{-\tilde{u}_2^2}(\Phi(\tilde{\boldsymbol{u}}_1') + \Phi(\tilde{\boldsymbol{u}}_2') - \Phi(\tilde{\boldsymbol{u}}_2) - \Phi(\tilde{\boldsymbol{u}}_1)), \tag{18}$$

The operation of $\tilde{\mathscr{D}}_K$ on the hydrodynamic fields can be read from (13)–(15). One obtains

$$\tilde{\mathscr{D}}_K \log n = K\left(\tilde{u}_i \frac{\partial \log n}{\partial \tilde{r}_i} - \left(\frac{3}{2\Theta}\right)^{1/2} \frac{\partial V_i}{\partial \tilde{r}_i}\right), \tag{19}$$

$$\tilde{\mathscr{D}}_K V_i = K\left(\tilde{u}_j \frac{\partial V_i}{\partial \tilde{r}_j} - \frac{1}{2}\left(\frac{2\Theta}{3}\right)^{1/2} \frac{\partial \log n}{\partial \tilde{r}_i} - \frac{1}{2}\left(\frac{2\Theta}{3}\right)^{1/2} \frac{\partial \log\Theta}{\partial \tilde{r}_i}\right), \tag{20}$$

and

$$\tilde{\mathscr{D}}_K \log\Theta = K\left(\tilde{u}_j \frac{\partial \log\Theta}{\partial \tilde{r}_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{1/2} \frac{\partial V_j}{\partial \tilde{r}_j}\right). \tag{21}$$

In deriving (20) and (21) we have employed the fact that $P_{ij} = \frac{1}{3} n\Theta\delta_{ij}$ to zeroth order in K and ϵ, and the heat flux, Q_i, is $O(K)$ to lowest order in K (hence its spatial derivatives are of higher order in K). Substitution of (19)–(21) in (17) results in

$$\tilde{\mathscr{L}}(\Phi_K) = 2K\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{1/2} \frac{\partial V_i}{\partial \tilde{r}_j} + K\left(\tilde{u}^2 - \tfrac{5}{2}\right)\tilde{u}_i \frac{\partial \log\Theta}{\partial \tilde{r}_i}, \tag{22}$$

where the overline denotes a symmetrized traceless tensor, i.e. $\overline{A_{ij}} \equiv \frac{1}{2}(A_{ij} + A_{ji}) - \frac{1}{3}A_{kk}\delta_{ij}$. Notice that (22) is identical to that obtained in the classical Chapman–Enskog expansion (of elastic systems) to first order in spatial gradients.

The isotropy of the operator $\tilde{\mathscr{L}}$ (Chapman & Cowling 1971 and Cercignani 1975) implies that the solution of (22) is of the form

$$\Phi_K(\tilde{\boldsymbol{u}}) = 2K\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{1/2} \frac{\partial V_i}{\partial \tilde{r}_j} + K\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \tfrac{5}{2}\right)\tilde{u}_i \frac{\partial \log\Theta}{\partial \tilde{r}_i}, \tag{23}$$

where $\hat{\Phi}_v(\tilde{u})$ and $\hat{\Phi}_c(\tilde{u})$ are functions of the (rescaled) speed $\tilde{u}$. It is common (Kogan 1969; Chapman & Cowling 1970; Harris 1971 and Cercignani 1975) to expand these functions in (truncated) series of Sonine polynomials. Here we prefer to expand them in sets of functions which obey the symmetry and asymptotic properties of $\hat{\Phi}_v$ and $\hat{\Phi}_c$ respectively (see Appendix A for details†) as we have

† Appendix A is available on request from the authors or the JFM Editorial Office.

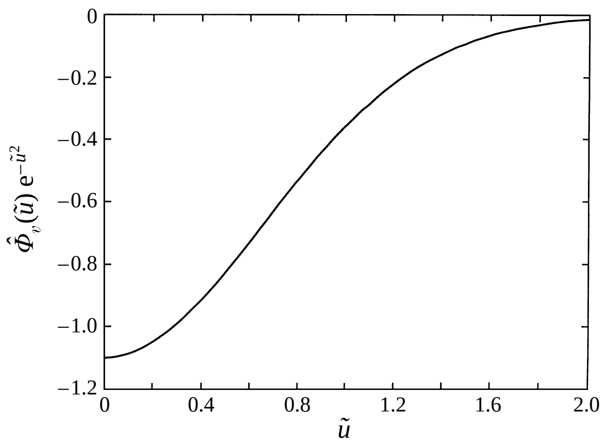

FIGURE 1. Plot of $\hat{\Phi}_v(\tilde{u})e^{-\tilde{u}^2}$ as a function of $\tilde{u}$ (cf. (23)).

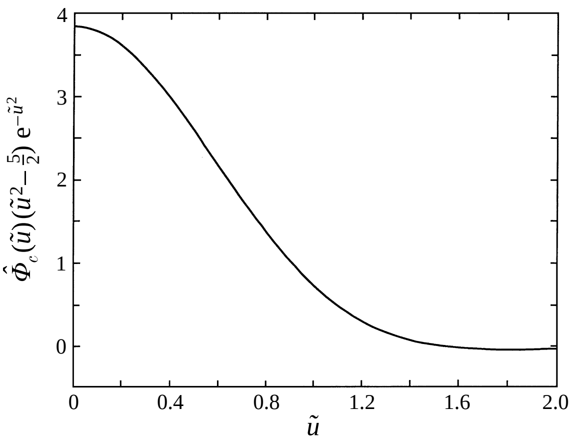

FIGURE 2. Plot of $\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)e^{-\tilde{u}^2}$ as a function of $\tilde{u}$ (cf. (23)).

done in the two-dimensional case (Sela *et al.* 1996). It turns out that the functions $\hat{\Phi}_v(\tilde{u})$ and $\hat{\Phi}_c(\tilde{u})$ are formally even in $\tilde{u}$ and they are both proportional to $1/\tilde{u}$ at large values of $\tilde{u}$ (a property that cannot be obeyed by a Sonine polynomial series). The functions $\hat{\Phi}_v$ and $\hat{\Phi}_c$, multiplied by $e^{-\tilde{u}^2}$ and $e^{-\tilde{u}^2}(\tilde{u}^2 - \frac{5}{2})$ respectively (for the sake of convenience), are depicted in figures 1 and 2, respectively.

Since the local equilibrium distribution function, f_0, is defined in such a way that the hydrodynamic fields are given by its appropriate moments, the contribution of the correction Φ to these moments should vanish, i.e. *Φ should be orthogonal with respect to the weight function, f_0, to the invariants of the (linearized) Boltzmann operator* (the eigenfunctions which correspond to zero eigenvalues): 1, $\tilde{\boldsymbol{u}}$ and $\tilde{u}^2$, whose respective averages are the density, the velocity and the temperature field. This orthogonality property should hold to all orders in perturbation theory (Kogan 1969; Chapman & Cowling 1970; Harris 1971 and Cercignani 1975); it is also the reason the (generalized) Chapman–Enskog expansion can be systematically carried out to all orders (cf. Appendix B) and it is repeatedly used below. Since the solution of equations of the type of (22) is determined up to the addition of an arbitrary combination of 1, $\tilde{\boldsymbol{u}}$ and $\tilde{u}^2$, it is the above orthogonality property that determines the required coefficients of these invariants. The orthogonality of the function Φ_K to $\tilde{\boldsymbol{u}}$ leads to the condition (on the basis of (23)): $\int_0^\infty \mathrm{d}\tilde{u}\tilde{u}^4 \mathrm{e}^{-\tilde{u}^2}\hat{\Phi}_c(\tilde{u})(\tilde{u}^2 - 5/2) = 0$. The other orthogonality conditions are identically satisfied by the right-hand side of (23). The determination of $\hat{\Phi}_v$ does not require the application of the orthogonality conditions.

The contribution of Φ_K to the stress tensor reads

$$P_{ij}^K = \int \mathrm{d}\boldsymbol{u} u_i u_j f_0(\tilde{u})\Phi_K = KnM_v \frac{16}{15\pi^{1/2}}\left(\frac{2\Theta}{3}\right)^{1/2}\overline{\frac{\partial V_i}{\partial \tilde{r}_j}}, \tag{24}$$

where M_v is given by $M_v = \int_0^\infty \mathrm{d}x x^6 \hat{\Phi}_v(x)\mathrm{e}^{-x^2} \approx -1.3224$ (the integration employs the numerically determined function $\hat{\Phi}_v$, see Appendix A). Notice that in (24) some variables are dimensionless and some are not; the integration is performed after $\tilde{u}$ is expressed in terms of u. A similar remark holds for all calculations below. Hence, one obtains

$$P_{ij}^K = -2\tilde{\mu}_0 n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}, \tag{25}$$

where $\tilde{\mu}_0 \approx 0.3249$. The subscript 0 denotes the fact that this coefficient is correct to zeroth order in ϵ. Similarly, the contribution of Φ_K to the heat flux is

$$Q_i^K = \frac{1}{2}\int \mathrm{d}\boldsymbol{u} u^2 u_i f_0(\tilde{u})\Phi_K = KnM_c \frac{2}{3\pi^{1/2}}\left(\frac{2\Theta}{3}\right)^{3/2}\frac{\partial \log\Theta}{\partial \tilde{r}_i}, \tag{26}$$

where: $M_c = \int_0^\infty \mathrm{d}x x^6 \left(x^2 - \frac{5}{2}\right)\hat{\Phi}_c(x)\mathrm{e}^{-x^2} \approx -2.003$. One thus obtains

$$Q_i^K = -\tilde{\kappa}_0 n\ell\Theta^{1/2}\frac{\partial\Theta}{\partial r_i}, \tag{27}$$

where $\tilde{\kappa}_0 \approx 0.4101$. These calculated values of the transport coefficients are in very close agreement with those calculated before for hard (smooth, elastic) spheres (Kogan 1969; Chapman & Cowling 1970; Harris 1971 and Cercignani 1975). Since, following (9), the energy sink term, Γ, has a prefactor $\epsilon \equiv 1 - e^2$, the function Φ_K should contribute a term which is $O(K\epsilon)$ to it. This term, denoted below by $\Gamma_{K\epsilon}$, can be computed by exploiting the invariance of the double integral in (9) to the exchange $\boldsymbol{u}_1 \leftrightarrow \boldsymbol{u}_2$:

$$\Gamma_{K\epsilon} = \frac{\epsilon\pi d^2}{4n}\int \mathrm{d}\boldsymbol{u}_1 \mathrm{d}\boldsymbol{u}_2 u_{12}^3 f_0(\tilde{u}_1) f_0(\tilde{u}_2)\Phi_K(\boldsymbol{u}_1). \tag{28}$$

Notice that if one considers first the integration over $\boldsymbol{u}_2$ then clearly $\int \mathrm{d}\boldsymbol{u}_2 u_{12}^3 f_0(\tilde{u}_2)$ is an isotropic function of u_1, i.e. one which depends on the speed $\tilde{u}_1$ alone. The integrand in (28) is therefore a product of $\Phi_K(\boldsymbol{u}_1)$ and an isotropic function. Thus, the form of Φ_K, cf. (23), implies, by symmetry considerations and the orthogonality conditions (which it must satisfy), that the integral in (28) vanishes, hence, $\Gamma_{K\epsilon} = 0$.

3.2. *Solution at $O(\epsilon)$*

In this subsection the Boltzmann equation is solved to first order in ϵ. The equation determining Φ_ϵ is obtained from (12) by expanding $\tilde{\mathscr{B}}(\tilde{f}, \tilde{f}, e)$ to first order in ϵ and retaining terms of $O(\epsilon)$. One obtains

$$\tilde{\mathscr{L}}(\Phi_\epsilon) = \tilde{\mathscr{D}}_\epsilon \log n + 2\left(\frac{3}{2\Theta}\right)^{1/2} \tilde{u}_i \tilde{\mathscr{D}}_\epsilon V_i + \left(\tilde{u}^2 - \tfrac{3}{2}\right) \tilde{\mathscr{D}}_\epsilon \log \Theta - \frac{\epsilon}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}} \mathrm{d}\tilde{\boldsymbol{u}}_2 (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}) \left(1 - \tfrac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) \mathrm{e}^{-\tilde{u}_2^2}. \tag{29}$$

The integral on the right-hand side of (29) is obtained from the expansion of $\tilde{\mathscr{B}}(\tilde{f}, \tilde{f}, e)$ to first order in ϵ, in (12), by utilizing the relation $\tilde{u}_1'^2 + \tilde{u}_2'^2 = \tilde{u}_1^2 + \tilde{u}_2^2 + \frac{1}{2}\epsilon(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 + O(\epsilon^2)$ in the exponent. Clearly (cf. (13)–(15)), $\tilde{\mathscr{D}}_\epsilon \log n = \tilde{\mathscr{D}}_\epsilon V_i = 0$ and: $\tilde{\mathscr{D}}_\epsilon \log \Theta = -\epsilon \tilde{\Gamma}_0$, where $\tilde{\Gamma}_0$ is the zeroth-order term in the expansion of $\tilde{\Gamma}$ (obtained by substituting $\tilde{f}_0$ for $\tilde{f}$ in (16)); its value is: $\tilde{\Gamma}_0 = \frac{2}{3}(2/\pi)^{1/2}$. Consequently, to $O(\epsilon)$: $\Gamma_\epsilon = (\epsilon/\ell)(16/27\pi)^{1/2}\Theta^{3/2}$. Carrying out the integral on the right-hand side of (29) the equation for Φ_ϵ assumes the form

$$\tilde{\mathscr{L}}(\Phi_\epsilon) = -\epsilon\left[\left(\frac{2}{\pi}\right)^{1/2}\left(\tfrac{2}{3}\tilde{u}^2 - 1\right) + \frac{3 - 2\tilde{u}^2}{8\pi^{1/2}}\mathrm{e}^{-\tilde{u}^2} + \frac{(5 + 4\tilde{u}^2 - 4\tilde{u}^4)\mathrm{erf}(\tilde{u})}{16\tilde{u}}\right]. \tag{30}$$

The right-hand side of (30) is orthogonal to the invariants 1, $\tilde{\boldsymbol{u}}$ and $\tilde{u}^2$, as one can easily verify by direct integration. Hence (30) is soluble. Appendix B presents a general proof that all equations that need to be solved in the framework of the perturbation theory employed in this article are soluble.

The isotropy of $\tilde{\mathscr{L}}$ implies that the solution of (30) assumes the form $\Phi_\epsilon(\tilde{\boldsymbol{u}}) = \epsilon\hat{\Phi}_e(\tilde{u})$ where $\hat{\Phi}_e$ is a function of the *speed* $\tilde{u}$. This isotropic correction has been missed in previous studies. In Appendix A it is shown that $\hat{\Phi}_e(\tilde{u})$ is formally even with respect to $\tilde{u}$ and that it is asymptotically (for $\tilde{u} \gg 1$) proportional to $\tilde{u}^2 \log \tilde{u}$. An expansion of $\hat{\Phi}_e$ in a set of functions obeying these symmetry and asymptotic properties is used in order to obtain a numerical solution of (30); to this (inhomogeneous) solution one must add a combination of the invariants to render it orthogonal to the invariants (cf. Appendix A). The function $\hat{\Phi}_e(\tilde{u})$ (multiplied by $\mathrm{e}^{-\tilde{u}^2}$) is depicted in figure 3. It is straightforward to deduce from the isotropy of Φ_ϵ and its orthogonality to the invariants that *it does not contribute to the stress tensor nor to the heat flux*. It only contributes a second order, in ϵ, term to Γ:

$$\Gamma_{\epsilon\epsilon} = \frac{\epsilon^2\pi d^2}{4n}\int \mathrm{d}\boldsymbol{u}_1 d\boldsymbol{u}_2 u_{12}^3 f_0(\tilde{u}_1) f_0(\tilde{u}_2)\hat{\Phi}_e(\tilde{u}_1). \tag{31}$$

The integrations over $\hat{\boldsymbol{u}}_1$ and $\hat{\boldsymbol{u}}_2$ in (31) are straightforward. The remaining double integral over u_1 and u_2 can be evaluated by numerical means. The result is: $\Gamma_{\epsilon\epsilon} \approx -0.0352\epsilon^2 n d^2 \Theta^{3/2}$. Another contribution to Γ stems from products of Φ_K and Φ_ϵ arising from the product of the expansions of the f factors in (9). It can be established by symmetry arguments, similar to those used to show that Φ_K does not contribute to Γ, that these products yield vanishing contributions to Γ (since Φ_ϵ is isotropic).

3.3. *Solution at $O(K\epsilon)$*

In this section the Boltzmann equation is solved at $O(K\epsilon)$. The correction to $\tilde{f}$ corresponding to this order, $\Phi_{K\epsilon}$, contributes terms which are first order in the spatial derivatives of the hydrodynamic fields (i.e. it belongs to the Navier–Stokes order), with coefficients that are proportional to ϵ. In order to calculate the contribution of $\Phi_{K\epsilon}$ to the transport coefficients one does not have to evaluate it. Following Chapman

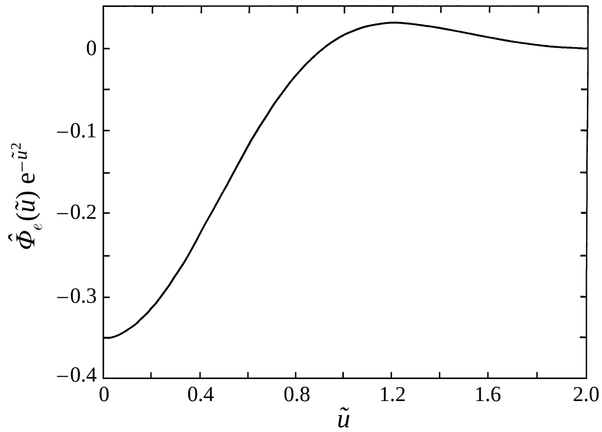

FIGURE 3. Plot of $\hat{\Phi}_e(\tilde{u})\,e^{-\tilde{u}^2}$ as a function of $\tilde{u}$ (cf. (30) and the text following it).

& Cowling (1970), the contribution of $\Phi_{K\epsilon}$ to the heat flux reads

$$Q_i^{K\epsilon} = \frac{1}{2}\int d\boldsymbol{u} u^2 u_i f_0 \Phi_{K\epsilon} = \frac{n}{2}\left(\frac{2\Theta}{3}\right)^{3/2}\int d\tilde{\boldsymbol{u}}\left(\tilde{u}^2 - \tfrac{5}{2}\right)\tilde{u}_i \tilde{f}_0 \Phi_{K\epsilon}. \tag{32}$$

The second equality in (32) results from the orthogonality of $\Phi_{K\epsilon}$ to $\tilde{u}_i$: $\int d\tilde{\boldsymbol{u}}\tilde{u}_i\tilde{f}_0\Phi_{K\epsilon} = 0$. Now, using the equality $\tilde{\mathscr{L}}[\hat{\Phi}_c(\tilde{u})(\tilde{u}^2 - \frac{5}{2})\tilde{u}_i] = (\tilde{u}^2 - \frac{5}{2})\tilde{u}_i$ (cf. (22), (23)), and exploiting the fact that $\tilde{\mathscr{L}}$ is self-adjoint, with f_0 serving as the weight function, one obtains that the correction to the heat flux due to $\Phi_{K\epsilon}$ is

$$Q_i^{K\epsilon} = \frac{n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{3/2}\int d\tilde{\boldsymbol{u}}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \tfrac{5}{2}\right)\tilde{u}_i e^{-\tilde{u}^2}\tilde{\mathscr{L}}(\Phi_{K\epsilon}). \tag{33}$$

A detailed calculation of the integral in (33) is presented in Appendix C. The result is

$$Q_i^{K\epsilon} = -\epsilon\tilde{\kappa}_1 n\ell\Theta^{1/2}\frac{\partial\Theta}{\partial r_i} - \epsilon\tilde{\tau}_1\ell\Theta^{3/2}\frac{\partial n}{\partial r_i}, \tag{34}$$

where $\tilde{\kappa}_1 \approx 0.1072$ and $\tilde{\tau}_1 \approx 0.2110$.

Similarly, the contribution of $\Phi_{K\epsilon}$ to the stress tensor is

$$P_{ij}^{K\epsilon} = \int d\boldsymbol{u} u_i u_j f_0 \Phi_{K\epsilon} = \frac{2n\Theta}{3}\int d\tilde{\boldsymbol{u}}\overline{\tilde{u}_i\tilde{u}_j} f_0 \Phi_{K\epsilon}. \tag{35}$$

The second equality results from the orthogonality of $\Phi_{K\epsilon}$ to $\tilde{u}^2$: $\int d\tilde{\boldsymbol{u}}\tilde{u}^2\tilde{f}_0\Phi_{K\epsilon} = 0$. Next, using the relation $\tilde{\mathscr{L}}[\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}] = \overline{\tilde{u}_i\tilde{u}_j}$, and exploiting the fact that $\tilde{\mathscr{L}}$ is self-adjoint, the correction to the stress tensor due to $\Phi_{K\epsilon}$ is

$$P_{ij}^{K\epsilon} = \frac{2n\Theta}{3\pi^{3/2}}\int d\tilde{\boldsymbol{u}}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}e^{-\tilde{u}^2}\tilde{\mathscr{L}}(\Phi_{K\epsilon}). \tag{36}$$

A detailed calculation of the above integral is presented in Appendix C. The result is

$$P_{ij}^{K\epsilon} = -2\epsilon\tilde{\mu}_1 n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}, \tag{37}$$

where $\tilde{\mu}_1 \approx 0.0576$. The tensorial structure of $\Phi_{K\epsilon}$ is similar to that of Φ_K hence its orthogonality properties imply that its contribution to Γ vanishes by symmetry considerations.

3.4. *Contributions of other second-order terms*

In order to complete the derivation of the constitutive relations to second order in the small parameters, ϵ and K, one has to consider Φ_{KK} and $\Phi_{\epsilon\epsilon}$. The term $\Phi_{\epsilon\epsilon}$ is of second order in ϵ and of zeroth order in the spatial gradients. Clearly, $\Phi_{\epsilon\epsilon}(\tilde{u})$ is an $O(\epsilon^2)$ isotropic function of the speed, $\tilde{u}$. Hence, the isotropy and the orthogonality conditions, which it must satisfy, imply that it does not contribute to the stress-tensor nor to the heat flux (for the same reasons that $\Phi_{\epsilon}(\tilde{u})$ does not contribute to them). Moreover, its contribution to Γ is $O(\epsilon^3)$ and it is therefore not considered here. The term Φ_{KK} is merely the perturbative contribution to f which is second order in the spatial derivatives (and zeroth order in ϵ). The contributions of Φ_{KK} to the dissipative fluxes, i.e. the stress tensor and the heat-flux vector, are the well-known (e.g. Kogan 1969) Burnett terms for elastic hard spheres.

In the inelastic case the Burnett term Φ_{KK} and the product of Navier–Stokes terms Φ_K (which result from the product of expansions of f in (9)) contribute to Γ terms which are $O(\epsilon K^2)$. These are formally of higher than second order in the small parameters but they are still Burnett terms (i.e. second order in the spatial derivatives). They contribute Burnett terms to the equation of motion of the energy field; moreover, this contribution is precisely of second order in the spatial derivatives since, unlike the dissipative fluxes whose gradients (or divergence) appear in the equations of motion, Γ appears as itself in the appropriate (energy) equation. The magnitude of this (Burnett) contribution to Γ is similar to that of the $O(\epsilon K)$ corrections to the leading Navier–Stokes terms to the equations of motion through the stress tensor and the heat-flux vector (second-order spatial derivatives and first order in ϵ). For these reasons we include the second order, in K, contributions to Γ in the the constitutive relations. Following a rescaling of (9), $\Gamma_{KK\epsilon}$ assumes the form

$$\Gamma_{KK\epsilon} = \frac{\epsilon}{8\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{3/2}(2I_1 + I_2), \tag{38}$$

where

$$I_1 \equiv \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \tilde{u}_{12}^3 \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \Phi_{KK}(\tilde{\boldsymbol{u}}_1), \tag{39}$$

and

$$I_2 \equiv \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \tilde{u}_{12}^3 \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \Phi_{K}(\tilde{\boldsymbol{u}}_1)\Phi_{K}(\tilde{\boldsymbol{u}}_2). \tag{40}$$

The factor 2 in (38) follows from the symmetry of the part of the integrand which includes $\Phi_{KK}(\tilde{\boldsymbol{u}}_1)$ or $\Phi_{KK}(\tilde{\boldsymbol{u}}_2)$, to the interchange of $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$. The integral I_1 can be written as follows:

$$I_1 = \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \chi(\tilde{u}_1) \Phi_{KK}(\tilde{\boldsymbol{u}}_1) \mathrm{e}^{-\tilde{u}_1^2}, \tag{41}$$

where

$$\chi(\tilde{u}_1) = \int \mathrm{d}\tilde{\boldsymbol{u}}_2 u_{12}^3 \mathrm{e}^{-\tilde{u}_2^2} = \pi\left(\left(\tilde{u}_1^2 + \tfrac{5}{2}\right)\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2}(3 + 12\tilde{u}_1^2 + 4\tilde{u}_1^4)}{4\tilde{u}_1}\mathrm{erf}(\tilde{u}_1)\right). \tag{42}$$

The function χ, given in (42), is not orthogonal to 1 and $\tilde{u}_1^2$. One may take advantage of the orthogonality of Φ_{KK} to the summational invariants of $\tilde{\mathscr{L}}$ to replace χ in (41) by $\overline{\chi} \equiv \chi - (4\pi/\sqrt{2})(1 + 2\tilde{u}_1^2)$. The latter function is orthogonal to all the summational invariants. Then, define $\overline{\eta}$ as the *unique* solution of the equation $\tilde{\mathscr{L}}(\overline{\eta}) = \overline{\chi}$, which is orthogonal to 1 and $\tilde{u}^2$. The function $\overline{\eta}(\tilde{u})$ is symmetric in $\tilde{u}$ and it has been evaluated

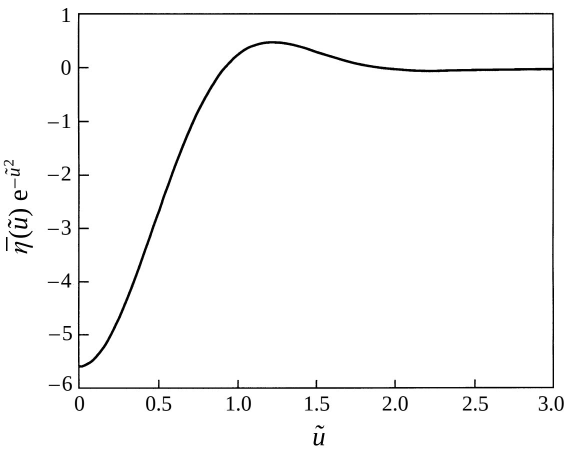

FIGURE 4. Plot of $\overline{\eta}(\tilde{u})\mathrm{e}^{-\tilde{u}^2}$ as a function of $\tilde{u}$.

in a similar way as $\hat{\Phi}_e$ (cf. Appendix A). The function $\overline{\eta}(\tilde{u})\mathrm{e}^{-\tilde{u}^2}$ is depicted in figure 4. Next, exploiting the fact that $\tilde{\mathscr{L}}$ is self-adjoint, I_1 can be represented as

$$I_1 = \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \overline{\eta}(\tilde{u}_1)\mathrm{e}^{-\tilde{u}_1^2} \tilde{\mathscr{L}}(\Phi_{KK}). \tag{43}$$

The calculation of I_1 and I_2 is straightforward but tedious. It is therefore relegated to Appendix E†. The resulting $O(\epsilon K^2)$ correction to Γ reads

$$\Gamma_{KK\epsilon} = \tilde{\rho}_1 \epsilon \ell \Theta^{1/2} \overline{\frac{\partial V_i}{\partial r_j}} \, \overline{\frac{\partial V_i}{\partial r_j}} + \tilde{\rho}_2 \frac{\epsilon \ell}{\Theta^{1/2}} \frac{\partial \Theta}{\partial r_i} \frac{\partial \Theta}{\partial r_i} + \tilde{\rho}_3 \frac{\epsilon \ell}{n \Theta^{1/2}} \frac{\partial (n\Theta)}{\partial r_i} \frac{\partial \Theta}{\partial r_i} + \tilde{\rho}_4 \epsilon \ell \Theta^{1/2} \frac{\partial^2 \Theta}{\partial r_i \partial r_i}, \tag{44}$$

where $\tilde{\rho}_1 \approx 0.1338$, $\tilde{\rho}_2 \approx 0.2444$, $\tilde{\rho}_3 \approx -0.0834$ and $\tilde{\rho}_4 \approx 0.0692$.

3.5. *Constitutive relations and normal stress difference*

In summary, to second order in K, and linear order in ϵ, the heat flux assumes the form

$$\begin{aligned} Q_i = &- \tilde{\kappa} n \ell \Theta^{1/2} \frac{\partial \Theta}{\partial r_i} - \tilde{\lambda} \ell \Theta^{3/2} \frac{\partial n}{\partial r_i} \\ &+ \tilde{\theta}_1 n \ell^2 \frac{\partial V_j}{\partial r_j} \frac{\partial \Theta}{\partial r_i} + \tilde{\theta}_2 n \ell^2 \left(\frac{2}{3} \frac{\partial}{\partial x_i} \left(\Theta \frac{\partial V_j}{\partial r_j} \right) + 2 \frac{\partial V_j}{\partial r_i} \frac{\partial \Theta}{\partial r_j} \right) \\ &+ \tilde{\theta}_3 \ell^2 \frac{\partial V_j}{\partial r_i} \frac{\partial (n\Theta)}{\partial r_j} + \tilde{\theta}_4 n \ell^2 \Theta \frac{\partial^2 V_j}{\partial r_i \partial r_j} + \tilde{\theta}_5 n \ell^2 \frac{\partial V_j}{\partial r_i} \frac{\partial \Theta}{\partial r_j}, \end{aligned} \tag{45}$$

where $\tilde{\kappa} \approx 0.4101 + 0.1072\epsilon + O(\epsilon^2)$, $\tilde{\lambda} \approx 0.2110\epsilon + O(\epsilon^2)$ and the values of the $\tilde{\theta}_i$ are (Kogan 1969 and Chapman & Cowling 1970): $\tilde{\theta}_1 \approx 1.2291$, $\tilde{\theta}_2 \approx -0.6146$, $\tilde{\theta}_3 \approx -0.3262$, $\tilde{\theta}_4 \approx 0.2552$, $\tilde{\theta}_5 \approx 2.6555$. Notice that the heat flux includes a term which is proportional to the density gradient. This term, which is mentioned in the introduction, is also proportional to ϵ and it does not exist in the standard Navier–Stokes theory; it is a consequence of inelasticity.

† Appendix E is available on request from the authors or the JFM Editorial Office.

The stress tensor, to second order in K and linear order in ϵ, reads

$$
\begin{aligned}
P_{ij} = {} & \tfrac{1}{3} n\Theta\delta_{ij} - 2\tilde{\mu} n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}} \\
& + \tilde{\varpi}_1 n\ell^2 \frac{\partial V_k}{\partial r_k}\overline{\frac{\partial V_i}{\partial r_j}} - \tilde{\varpi}_2 n\ell^2 \left(\frac{1}{3}\overline{\frac{\partial}{\partial r_i}\left(\frac{1}{n}\frac{\partial(n\Theta)}{\partial r_j}\right)} + \overline{\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}} + 2\overline{\overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial V_k}{\partial r_j}} \right) \\
& + \tilde{\varpi}_3 n\ell^2 \overline{\frac{\partial^2\Theta}{\partial r_i \partial r_j}} + \tilde{\varpi}_4 \frac{\ell^2}{\Theta}\overline{\frac{\partial(n\Theta)}{\partial r_i}\frac{\partial\Theta}{\partial r_j}} + \tilde{\varpi}_5 \frac{n\ell^2}{\Theta}\overline{\frac{\partial\Theta}{\partial r_i}\frac{\partial\Theta}{\partial r_j}} + \tilde{\varpi}_6 n\ell^2 \overline{\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}},
\end{aligned}
\tag{46}
$$

where $\tilde{\mu} \approx 0.3249 + 0.0576\epsilon + O(\epsilon^2)$ and the values of the $\tilde{\varpi}_i$ are (Kogan 1969 and Chapman & Cowling 1970): $\tilde{\varpi}_1 \approx 1.2845$, $\tilde{\varpi}_2 \approx 0.6422$, $\tilde{\varpi}_3 \approx 0.2552$, $\tilde{\varpi}_4 \approx 0.0719$, $\tilde{\varpi}_5 \approx 0.0231$, $\tilde{\varpi}_6 \approx 2.3510$.

The inelastic dissipation term, Γ, to second order in K and up to second order in ϵ, reads

$$
\Gamma = \frac{\tilde{\delta}}{\ell}\Theta^{\frac{3}{2}} + \tilde{\rho}_1 \epsilon\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} + \tilde{\rho}_2 \frac{\epsilon\ell}{\Theta^{1/2}}\frac{\partial\Theta}{\partial r_i}\frac{\partial\Theta}{\partial r_i} + \tilde{\rho}_3 \frac{\epsilon\ell}{n\Theta^{1/2}}\frac{\partial(n\Theta)}{\partial r_i}\frac{\partial\Theta}{\partial r_i} + \tilde{\rho}_4 \epsilon\ell\Theta^{1/2}\frac{\partial^2\Theta}{\partial r_i \partial r_i},
\tag{47}
$$

where $\tilde{\delta} \approx (16/27\pi)\epsilon - 0.0112\epsilon^2$, $\tilde{\rho}_1 \approx 0.1338$, $\tilde{\rho}_2 \approx 0.2444$, $\tilde{\rho}_3 \approx -0.0834$ and $\tilde{\rho}_4 \approx 0.0692$. We reiterate that Γ is proportional to $(1/\ell)$ (which follows from (9) after non-dimensionalizing the integrand). Hence, to leading order in K and ϵ (i.e. K^0 and ϵ^1) its dependence on ℓ is given by $(1/\ell)$. The next non-vanishing contribution to Γ is of second order in the Knudsen number and it is proportional to ℓ. This property, which is specific to inelastic systems, indicates that (unlike in elastic systems) one cannot deduce the Knudsen order of a term in the hydrodynamic equations by counting the power of ℓ in its prefactor; instead one must consider the appropriate order in the expansion of f or count spatial derivatives as explained below. As mentioned above, the time derivatives of the mass, momentum and energy density fields are respectively divergences of corresponding fluxes, except the latter field whose equation of motion includes the term Γ which is not a divergence of a flux. As a result, a term in the equations of motion which contains n spatial derivatives is of $O(K^{n-1})$ unless this term belongs to Γ, in which case it is $O(K^n)$ (the order in ϵ notwithstanding).

The anisotropy of the stress tensor, i.e. the normal stress difference, is a Burnett effect (as in the case of a two-dimensional shear flow). This fact is observed upon substituting a simple-shear flow field $\boldsymbol{V} = \gamma y\hat{\boldsymbol{x}}$, in (46). The resulting diagonal components of the stress tensor are: $P_{xx} = \frac{1}{3}n\Theta + \frac{1}{12}(\tilde{\varpi}_6 + 4\tilde{\varpi}_2)n\ell^2\gamma^2$, $P_{yy} = \frac{1}{3}n\Theta + \frac{1}{12}(\tilde{\varpi}_6 - 8\tilde{\varpi}_2)n\ell^2\gamma^2$ and $P_{zz} = \frac{1}{3}n\Theta + \frac{1}{6}(\tilde{\varpi}_6 - 2\tilde{\varpi}_2)n\ell^2\gamma^2$.

The normal stress difference (normalized by the pressure $P \equiv \frac{1}{3}n\Theta$), under steady-state conditions, is obtained by using the above components of the stress tensor together with the requirement that the heating and cooling rates are equal (this is read from (8)):

$$
\frac{P_{xx} - P_{yy}}{P} = \frac{6\tilde{\varpi}_2\tilde{\delta}}{4\tilde{\mu} - \tilde{\rho}_1\epsilon},
\tag{48}
$$

where the numerical constants on the right-hand side are given in the above. The normal stress differences (as obtained from (48)) for values of $e = 0.8$ and $e = 0.6$ are ≈ 0.45 and ≈ 0.88 respectively. These values compare very well with numerical

(Molecular Dynamics) results (Walton & Braun 1986) ≈ 0.42 and ≈ 0.86 respectively (for a volume fraction $v = 0.025$). Our predictions are still slightly higher than those measured by Walton & Braun since the Boltzmann theory is valid in the limit $v \to 0$ while their calculations are performed at finite (even if small) values of the number density (or volume fraction), and the normal stress difference decreases as the number density increases (cf. Walton & Braun 1986). As a matter of fact, one may consider the normal stress differences as a measurable manifestation of the Burnett terms in sheared granular flows (Goldhirsch & Sela 1996). Clearly, our results also imply that the matrix of correlations of the velocity fluctuations is non-diagonal and that its diagonal entries are different from each other. The inverse of this matrix is often referred to as the 'anisotropic temperature'; this anisotropy is clearly a result of our theory, the normal stress difference being responsible for the 'diagonal anisotropy', to use a similar terminology.

Possible physical setups in which this effect can be measured are Couette and Taylor–Couette systems as well as vibrated granular systems which are fluidized near the base surface of the container. It is possible that it can also be observed in the fluidized phase near the floor of a chute but this may be too hard a measurement to perform.

3.6. *Comparison to previous theories*

There are two major methods by which hydrodynamic equations for granular fluids have been derived on the basis of the Boltzmann equation. In one of them (Lun *et al.* 1984; Jenkins & Richman 1988; Boyle & Massoudi 1990; Lun 1991 and Goldshtein & Shapiro 1995) the single particle distribution function is conjectured to have the form of a perturbed local Maxwellian, where the perturbations are the gradients of the field variables multiplied by numerical coefficients (that are determined by requiring consistency with the Enskog equations or Maxwell's conservation integrals). The second method is based on the Grad expansion (Jenkins & Richman 1985*a,b*). Both methods yield similar constitutive relations. The above-mentioned (as well as other) studies did not consider Burnett terms but they included Enskog corrections in their calculations in order to render the theory valid at moderately high densities. Hence, a comparison of our results for the constitutive relations to those of previous studies is possible only at the Navier–Stokes level (i.e. to first order in the gradients of the hydrodynamic fields) and to lowest (zeroth) order in the volume fraction, v. We shall also restrict the comparison to the linear order in the degree of inelasticity, ϵ. To the above orders in ϵ, K and v, the result obtained before for the pressure tensor is (after translating the granular temperature T used by Lun *et al.* (1984) and Jenkins & Richman (1985*a*) to our notation $\Theta = 3T$):

$$P^*_{ij} = \tfrac{1}{3} n\Theta\delta_{ij} - 2\tilde{\mu}^* n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}, \tag{49}$$

where, $\tilde{\mu}^* = \frac{5}{16}(\pi/3)^{1/2}(1-\frac{1}{2}\epsilon^2)$. To zeroth order, in ϵ, this formula for $\tilde{\mu}^*$, agrees with our result. The next non-vanishing order, in ϵ, in the above expression for the viscosity, is $O(\epsilon^2)$ in contrast with our finding that it is $O(\epsilon)$. In the same approximation the heat-flux vector, following Lun *et al.* (1984) and Jenkins & Richman (1985*a*) reads (e.g. (4.19) in Lun *et al.* 1984)

$$Q^*_i = -\tilde{\kappa}^* n\ell\Theta^{1/2}\frac{\partial\Theta}{\partial r_i} - \tilde{\lambda}^*\ell\Theta^{3/2}\frac{\partial n}{\partial r_i}, \tag{50}$$

where, $\tilde{\kappa}^* = \frac{25}{64}(\pi/3)^{1/2}(1-\frac{25}{32}\epsilon)$ and $\tilde{\lambda}^* = -\frac{15}{32}(\pi/3)^{1/2}v\epsilon$. Although the above expression for the heat flux contains a term which is proportional to the density gradient, this term is $O(n)$ and it stems from the Enskog correction to the Boltzmann equation; it vanishes in the Boltzmann limit, in contrast with (45) above, in which a term proportional to $\epsilon\nabla \log n$ exists. In addition, the sign of the $O(\epsilon)$, term in the expression for $\tilde{\kappa}^*$ obtained by Lun *et al.* (1984) and Jenkins & Richman (1985*a*) is opposite to the sign obtained in the present paper. Another difference exists in the form of the energy sink term. The result obtained by them (e.g. (4.23) in Lun *et al.* 1984) is

$$\Gamma^* = \frac{\tilde{\delta}^*}{\ell}\Theta^{3/2}, \tag{51}$$

where $\tilde{\delta}^* = (16/27\pi)^{1/2}\epsilon + O\ (\epsilon^3)$, whereas the theory developed here results in an $O(\epsilon^2)$ correction to the leading term (the latter is $O(\epsilon)$ in both theories) and terms of $O(\epsilon K^2)$ which contribute dissipation terms of the same magnitude (and order) as the $O(\epsilon)$ corrections to the stress and heat flux.

The qualitative differences between the theories developed by Lun *et al.* (1984) and Jenkins & Richman (1985*a*) and our theory are not due to the differences in the respective approaches since both methods allow for general ϵ dependence and corrections to the Maxwellian distribution function. A careful examination of the analyses performed by Lun *et al.* (1984) and Jenkins & Richman (1985*a*), which are practically equivalent to each other, reveals the reason for the discrepancies, described in the above: these theories do not take into account the quasi-microscopic time scale for the decay of the granular temperature, τ/ϵ, where τ is the mean free time. Since the transport coefficients attain their respective asymptotic values on a time scale of order τ (e.g. (52),(53) in Jenkins & Richman 1985*a*), neglect of the above time scale for the temperature gives rise to differences at $O(\epsilon)$ between our results and previous ones. This should not be taken as criticism of the pioneering works cited above since they did not claim correctness beyond the leading order in ϵ. In the next section we perform an analysis which is similar to that presented in Jenkins & Richman (1985*a*) *while taking into account the fast time dependence of the granular temperature*. The result is transport coefficients which are in agreement with those derived by the present (generalized) Chapman–Enskog expansion. Some minor (quantitative) differences between the results obtained here for the transport coefficients and those obtained by employing Grad's method are due to the fact that the standard application of the latter method does not include the isotropic correction Φ_ϵ and the functions $\hat{\Phi}$ (of the speed) are represented by effective constants (since Grad's method is basically a fit to f).

4. Application of Grad's method to granular fluids

In this section it is demonstrated that Grad's method of moments (Grad 1949) is applicable to the description of granular fluids provided one takes into account the fact that there is a microscopic time scale characterizing the dynamics of the granular temperature. This time scale becomes macroscopic in the elastic limit (see below).

Grad's method involves an expansion of the single-particle distribution function around a local Maxwellian distribution. The (multiplicative) correction to the Maxwellian is usually assumed to be of the form of a series of (orthogonal) polynomials in the fluctuating velocity (components), each of which has a time-dependent prefactor (also known as a 'moment' for obvious reasons) which is calculated as part of the Grad method. Let m_β (where β usually represents a tensorial index) denote a typical 'moment'. This quantity can be shown to satisfy an equation of motion of the form (Kogan 1969)

$$\frac{\partial m_\beta}{\partial t} + \frac{1}{\tau} m_\beta + A_\beta = 0, \tag{52}$$

where $\tau \propto \ell/\Theta^{1/2}$ is proportional to the (microscopic) mean free time between collisions. The term A_β represents 'slow' variables (which are assumed to vary on hydrodynamic time scales). Upon formally solving (52) and noting that τ is by itself a time-dependent entity (e.g. through its dependence on Θ) one obtains the following (asymptotic) expansion for m_β:

$$m_\beta(t) = \left[m_\beta + \tau A_\beta - \tau \frac{\partial}{\partial t}(\tau A_\beta) + ... \right]_{t=0} \exp\left(- \int_0^t \frac{dt'}{\tau(t')} \right) - \left(\tau A_\beta - \tau \frac{\partial}{\partial t}(\tau A_\beta) + ... \right). \tag{53}$$

For 'asymptotically' long times ($t \gg \tau$) one obtains from (53)

$$m_\beta = -\tau A_\beta + \tau \frac{\partial}{\partial t}(\tau A_\beta) + \ldots, \tag{54}$$

i.e. the value of $m_\beta(t)$ depends on the value of $A_\beta(t)$ and its time derivatives (at time t). In the case of fluids whose constituents collide elastically, the first term on the right-hand side of (54) yields the Navier–Stokes constitutive relations and the second term begets the Burnett correction (the reason is that the action of a time derivative on a hydrodynamic field equals the divergence of an expression, thereby producing a term which is one order higher in the gradients). This is not the case for granular fluids since the time derivative of the temperature field, Θ, includes a (dissipation) term which, to leading order in K, contains no spatial derivatives. More precisely, since $\tau \propto \ell/\Theta^{1/2}$ and $\partial\Theta/\partial t \propto -(\epsilon/\tau)\Theta$, it follows that $\partial\Theta/\partial t \propto -\epsilon$ to leading order in the Knudsen number. It follows that each of the higher orders in (54) contributes terms which are $O(K)$, i.e. of Navier–Stokes order, though of increasingly higher orders in ϵ. In addition, if A_β is chosen to be the temperature field, it contributes in a similar manner to the Navier–Stokes order. The significance of the above observation can perhaps be better appreciated by noting that τ/ϵ is the quasi-microscopic time scale characterizing rate of decay of the temperature; this time scale must be taken into account, as demonstrated below. To reiterate, the second term in the expansion on the right-hand side of (54) contributes corrections to the Navier–Stokes constitutive relations which are $O(\epsilon)$, and the next order terms (those which are not explicitly presented in (54)) contribute further corrections to the Navier–Stokes order (i.e. linear in the gradients) which are $O(\epsilon^2)$ and higher. Therefore, unlike in the elastic case, every term in the asymptotic series in (54) contributes to the Navier–Stokes order.

Below we make use of the above observations in order to obtain constitutive relations for granular flows by using Grad's method of moments. To this end we use the results obtained by Lun *et al.* (1984) and Jenkins & Richman (1985*a*). We only consider a restricted version of their results since the present article is concerned with near elastic collisions and the Boltzmann level of description of the dynamics: (i) only the terms which are of zeroth order in the number density are considered; (ii) the functions $\Theta_i(\psi)$ are neglected (cf. (23) in Jenkins & Richman 1985*a*); (iii) only contributions up to first order in ϵ are taken into account; (iv) only the Navier–Stokes level of description (i.e. terms which are first order in spatial gradients) is considered.

Following Jenkins & Richman (1985*a*), the stress tensor is determined by the moments a_{ij}. Under the above-stated restrictions the equation satisfied by a_{ij} is

$$\frac{\mathrm{d}a_{ij}}{\mathrm{d}t} + \frac{1}{\tau_1} a_{ij} + 2T\hat{D}_{ij} = 0, \tag{55}$$

where $\tau_1 = 5/(16\pi^{1/2} n d^2 T^{1/2})$ is proportional to the mean free time (d is the diameter of the spheres) and $\hat{D}_{ij}$ is the symmetrized, traceless, strain-rate tensor. Notice that the granular temperature denoted by T in Jenkins & Richman (1985*a*) is related to Θ by $\Theta = 3T$. It is standardly argued in applications of Grad's method (cf. also Jenkins & Richman 1985*a*) that it takes only few collisions per particle for a_{ij} to saturate to the value (to first order in ϵ)

$$a^*_{ij} = -2\tau_1 T\hat{D}_{ij} = -\tfrac{5}{8}\pi^{1/2}(1 + O(\epsilon^2))\ell T^{1/2}\hat{D}_{ij}. \tag{56}$$

This approach is equivalent to truncating (54) at the first term. However, as we have shown in the above, one has to take the second term into account as well in order to obtain the Navier–Stokes constitutive relations, correct to first order in ϵ. One, therefore, obtains

$$a_{ij} = -2\tau_1 T\hat{D}_{ij} + \tau_1 \frac{\partial}{\partial t}(2\tau_1 T\hat{D}_{ij}). \tag{57}$$

Using the definition of τ_1 in the above and the $O(\epsilon)$ equation relation for the decay rate of the granular temperature (cf. Jenkins & Richman 1985*a*): $\mathrm{d}T/\mathrm{d}t = -(4/3\pi^{1/2})(\epsilon/\ell)T^{3/2}$, one obtains, at the Navier–Stokes order

$$a_{ij} = -\tfrac{5}{8}\pi^{1/2}\left(1 + \tfrac{5}{24}\epsilon + O(\epsilon^2)\right)\ell T^{1/2}\hat{D}_{ij}, \tag{58}$$

which implies that the deviatoric stress tensor assumes the form $P_{ij} = -2\mu\hat{D}_{ij}$, where μ is the viscosity:

$$\mu = \frac{5\pi^{1/2}}{16}\left(1 + \tfrac{5}{24}\epsilon + O(\epsilon^2)\right) n\ell T^{1/2}. \tag{59}$$

The heat flux is determined by a_{ijj} (cf. Jenkins & Richman 1985*a*). Under the above restrictions, the equation satisfied by a_{ijj} reads

$$\frac{\mathrm{d}a_{ijj}}{\mathrm{d}t} + \frac{1}{\tau_2} a_{ijj} + 5T\frac{\partial T}{\partial r_i} = 0, \tag{60}$$

where $\tau_2 = 15/(32\pi^{1/2}(1 + 25\epsilon/32) n d^2 T^{1/2})$. In Jenkins & Richman (1985*a*), similar considerations to those leading to (56), have resulted in the following solution of (60):

$$a^*_{ijj} = -5\tau_2 T\frac{\partial T}{\partial r_i} = -\frac{75\pi^{1/2}}{32}\left(1 - \frac{25\epsilon}{32} + O(\epsilon^2)\right)\ell T^{1/2}\frac{\partial T}{\partial r_i}. \tag{61}$$

As shown above, the correct dependence on ϵ (to linear order in ϵ) can be obtained by taking into account the first two terms in the expansion, (54):

$$\begin{aligned} a_{ijj} &= -5\tau_2 T\frac{\partial T}{\partial r_i} + \tau_2\frac{\partial}{\partial t}\left(5\tau_2 T\frac{\partial T}{\partial r_i}\right) \\ &= -\frac{75\pi^{1/2}}{32}\left[\left(1 + \frac{15}{32}\epsilon\right)\ell T^{1/2}\frac{\partial T}{\partial r_i} + \frac{5}{8}\epsilon\ell\frac{T^{3/2}}{n}\frac{\partial n}{\partial r_i}\right]. \end{aligned} \tag{62}$$

It follows that the heat-flux vector assumes the form

$$Q_i = -\kappa \frac{\partial T}{\partial r_i} - \lambda \frac{\partial n}{\partial r_i}$$

where

$$\kappa = \frac{75\pi^{1/2}}{64}\left(1 + \frac{15}{32}\epsilon + O(\epsilon^2)\right) n\ell T^{1/2} \quad \text{and} \quad \lambda = \frac{375\pi^{1/2}}{256}(\epsilon + O(\epsilon^2))\, \ell T^{3/2}.$$

The transport coefficients derived by using the generalized Chapman–Enskog expansion read, following an appropriate translation of variables, as follows:

$$\mu_{CE} \approx \sqrt{3}(0.3249 + 0.0576\epsilon + O(\epsilon^2))\, n\ell T^{1/2}, \tag{63}$$

$$\kappa_{CE} \approx 3\sqrt{3}(0.4101 + 0.1072\epsilon + O(\epsilon^2))\, n\ell T^{1/2}, \tag{64}$$

$$\lambda_{CE} \approx 3\sqrt{3}(0.2110\epsilon + O(\epsilon^2))\, \ell T^{3/2}. \tag{65}$$

It is easy to check that the latter transport coefficients are in good agreement with those derived by using the 'correct' Grad method. The main quantitative difference is due to the fact that the standard application of Grad's method does not include isotropic corrections to the Maxwellian distribution (which automatically arise in the CE approach; these corrections can be included in the Grad expansion). In addition, our calculations include accurate determinations of the corrections to the Maxwellian distribution by (numerically) solving the appropriate integral equations (rather than using effective constants).

5. Conclusion

We have developed a generalization of the Chapman–Enskog expansion for analysing the Boltzmann equation pertaining to a monodisperse collection of smooth spheres and we have carried out this expansion to Burnett order and derived constitutive relations to this order. These relations are different from previously derived constitutive relations and this difference has been explained. In principle, generalizations of this work to include tangential restitution (in preparation) and polydispersivity are algebraically tedious but they do not seem to require any novel ideas. The study of non-spherical particles, which is of importance to practical applications, seems to be possible as well, at least in the case of ellipsoidal particles. A generalization of this work (which is formally limited to near-elastic collisions) to strongly inelastic systems is of importance. Judging by the small prefactors of the higher-order terms in the Chapman–Enskog perturbative series it seems that the expansion in powers of ϵ is suitable for obtaining constitutive relations for strongly inelastic systems in spite of the fact that the theory is formally limited to near-elastic systems. On the other hand, the treating of high shear rates or high temperature gradients poses a true challenge, as it does in the realm of elastic systems. One of the difficulties encountered when attempting to tackle the latter problem stems from the fact that the Burnett equations (though useful for steady states) are mathematically ill-posed (Bobylev 1984, and references therein). While this problem is known to exist in the realm of molecular gases it is only of importance there in extreme cases such as those that occur in strong shocks (Fiscko & Chapman 1989). In contrast, in the field of granular systems, the shear rates are, practically always, relatively 'high'; consider e.g. the steady-state temperature in a sheared granular system: $T \propto \gamma^2\ell^2/\epsilon$ (as mentioned above, this relation follows straightforwardly from the equations of motion). The ratio $\gamma\ell/T^{1/2}$ is practically always $O(1)$ unless the system is nearly elastic and thus the change of the macroscopic velocity over the scale of a mean free path is of the order of the thermal speed. Thus, 'far from elastic' and 'strongly sheared' are intimately related in granular flows. It is possible that some resummation techniques (as proposed e.g. by Rosenau 1989), developed for molecular systems, may be of use in obtaining well-posed equations of motion for granular systems, which are not limited to the Navier–Stokes order. This, however, remains to be seen.

One of us (I.G.) gratefully acknowledges partial support from the US-Israel Binational Science Foundation and the National Science Foundation.

Appendix A is available on request from the authors or the JFM Editorial Office.

Appendix B. Proof of solubility

In this Appendix it is shown that the generalized Chapman–Enskog expansion, developed in this work, results in soluble equations to all orders in the expansion. To this end we consider (10). This equation can be written as

$$\tilde{f}_0\tilde{\mathscr{L}}(\Phi) = \tilde{\mathscr{D}}\tilde{f} + \tilde{f}\tilde{\mathscr{D}}\left(\log n - \tfrac{3}{2}\log\Theta\right) - \tfrac{1}{2}\tilde{f}_0\tilde{\Omega}(\Phi,\Phi) - (\tilde{\mathscr{B}}(\tilde{f},\tilde{f},e) - \tilde{\mathscr{B}}_{el}(\tilde{f},\tilde{f})), \quad \text{(B 1)}$$

where $\tilde{\Omega}$ is defined in (C 3) of Appendix C and $\tilde{\mathscr{B}}_{el}(\tilde{f},\tilde{f}) \equiv \tilde{\mathscr{B}}(\tilde{f},\tilde{f},e=1)$ is the collision integral corresponding to elastic collisions. Equation (B 1) is soluble only if its right-hand side is orthogonal to all the summational invariants of $\tilde{\mathscr{L}}$: 1, $\tilde{\boldsymbol{u}}$ and $\tilde{u}^2$. The integral over $\boldsymbol{u}_1$ of the first term on the right-hand side of (B 1) times any of the summational invariants can be carried out as follows. Consider the integration involving the first summational invariant:

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\tilde{\mathscr{D}}\tilde{f} = \ell\left(\frac{3}{2\Theta}\right)^2 \int \mathrm{d}\boldsymbol{v}\left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial r_i}\right)\left(\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{3/2} f\right). \quad \text{(B 2)}$$

Clearly, the partial derivatives with respect to t and r_i can be placed in front of the integral. Then, using the definition of the field variables given in (3)–(5) and (6)–(8), one obtains

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\tilde{\mathscr{D}}\tilde{f} = -K\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{3/2}\left(P'_{ij}\frac{\partial V_i}{\partial \tilde{r}_j} + \frac{\partial Q_i}{\partial \tilde{r}_i}\right) - \tfrac{3}{2}\epsilon\tilde{\Gamma} \quad \text{(B 3)}$$

where $P'_{ij} \equiv P_{ij} - \frac{1}{3}n\Theta\delta_{ij}$ is the deviatoric stress tensor. In a similar way one obtains

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\tilde{u}_i\tilde{\mathscr{D}}\tilde{f} = K\frac{3}{2n\Theta}P_{ij}\frac{\partial}{\partial \tilde{r}_j}\left(\tfrac{3}{2}\log\Theta - \log n\right) \quad \text{(B 4)}$$

and

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\tilde{u}^2\tilde{\mathscr{D}}\tilde{f} = K\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{3/2}\left(Q_i\frac{\partial \log n}{\partial \tilde{r}_i} - \tfrac{3}{2}Q_i\frac{\partial \log\Theta}{\partial \tilde{r}_i} + \tfrac{3}{2}P'_{ij}\frac{\partial V_i}{\partial \tilde{r}_j} + \frac{3}{2}\frac{\partial Q_i}{\partial \tilde{r}_i}\right) + \tfrac{9}{4}\epsilon\tilde{\Gamma}. \quad \text{(B 5)}$$

The integral of the second term on the right-hand side of (B 1) times any of the summational invariants is performed by using the following relation which follows directly from (13)–(15):

$$\tilde{\mathscr{D}}\left(\log n - \tfrac{3}{2}\log\Theta\right) = K\left(\tilde{u}_i\frac{\partial \log n}{\partial \tilde{r}_i} - \tfrac{3}{2}\tilde{u}_i\frac{\partial \log\Theta}{\partial \tilde{r}_i} + \frac{2}{n}\left(\frac{3}{2\Theta}\right)^{3/2} P'_{ij}\frac{\partial V_i}{\partial \tilde{r}_j} + \frac{2}{n}\left(\frac{3}{2\Theta}\right)^{3/2}\frac{\partial Q_i}{\partial \tilde{r}_i} + \tfrac{3}{2}\epsilon\tilde{\Gamma}\right). \tag{B 6}$$

One obtains

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\tilde{f}\tilde{\mathscr{D}}\left(\log n - \tfrac{3}{2}\log\Theta\right) = K\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{3/2}\left(P'_{ij}\frac{\partial V_i}{\partial \tilde{r}_j} + \frac{\partial Q_i}{\partial \tilde{r}_i}\right) + \tfrac{3}{2}\epsilon\tilde{\Gamma}, \tag{B 7}$$

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\tilde{u}_i\tilde{f}\tilde{\mathscr{D}}\left(\log n - \tfrac{3}{2}\log\Theta\right) = -K\frac{3}{2n\Theta}P_{ij}\frac{\partial}{\partial \tilde{r}_j}\left(\tfrac{3}{2}\log\Theta - \log n\right), \tag{B 8}$$

and

$$\int \mathrm{d}\tilde{\mathbf{u}}\tilde{u}^2\tilde{f}\tilde{\mathscr{D}}\left(\log n - \tfrac{3}{2}\log\Theta\right) = K\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{3/2}\left(-Q_i\frac{\partial \log n}{\partial \tilde{r}_i} + \frac{3}{2}Q_i\frac{\partial \log\Theta}{\partial \tilde{r}_i} - \tfrac{3}{2}P'_{ij}\frac{\partial V_i}{\partial \tilde{r}_j} - \tfrac{3}{2}\frac{\partial Q_i}{\partial \tilde{r}_i}\right) - \frac{15}{4}\epsilon\tilde{\Gamma}. \tag{B 9}$$

The integral of the third term on the right-hand side of (B 1) times any of the summational invariants vanishes due to the symmetry properties which follow from the fact that the operator $\tilde{\Omega}$ is defined with an elastic velocity transformation ((C 3) in Appendix C). Finally consider the integral of the fourth term on the right-hand side of (B 1) times the summational invariants. Clearly, the contribution of $\tilde{\mathscr{B}}_{el}$ to any of the integrals vanishes due to the conservation of mass, momentum and energy in elastic collisions. Moreover, the contributions of $\tilde{\mathscr{B}}$ to the integrals involving 1 and $\tilde{u}_i$ vanish. The integral involving $\tilde{u}^2$ is

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\tilde{u}^2(\tilde{\mathscr{B}}(\tilde{f},\tilde{f},e) - \tilde{\mathscr{B}}_{el}(\tilde{f},\tilde{f})) = -\tfrac{3}{2}\epsilon\tilde{\Gamma}. \tag{B 10}$$

Combining all of the results it follows that the right-hand side of (B 1) is orthogonal to all the summational invariants of $\tilde{\mathscr{L}}$.

Appendix C. Constitutive relations at $O(K\epsilon)$

In this Appendix, a detailed derivation of the constitutive relations at $O(K\epsilon)$ is presented. At this order (12) reads

$$\begin{aligned}
&\tilde{\mathscr{D}}_{K\epsilon}\log n + 2\left(\frac{3}{2\Theta}\right)^{1/2}\tilde{u}_i\tilde{\mathscr{D}}_{K\epsilon}V_i + \left(\tilde{u}^2 - \tfrac{3}{2}\right)\tilde{\mathscr{D}}_{K\epsilon}\log\Theta \\
&+ \Phi_K\left(\tilde{\mathscr{D}}_{\epsilon}\log n + 2\left(\frac{3}{2\Theta}\right)^{1/2}\tilde{u}_i\tilde{\mathscr{D}}_{\epsilon}V_i + \left(\tilde{u}^2 - \tfrac{3}{2}\right)\tilde{\mathscr{D}}_{\epsilon}\log\Theta\right) \\
&+ \Phi_\epsilon\left(\tilde{\mathscr{D}}_K\log n + 2\left(\frac{3}{2\Theta}\right)^{1/2}\tilde{u}_i\tilde{\mathscr{D}}_K V_i + \left(\tilde{u}^2 - \tfrac{3}{2}\right)\tilde{\mathscr{D}}_K\log\Theta\right) + \tilde{\mathscr{D}}_K\Phi_\epsilon + \tilde{\mathscr{D}}_\epsilon\Phi_K \\
&= \tilde{\mathscr{L}}(\Phi_{K\epsilon}) + \epsilon\tilde{\Xi}(\Phi_K) + \epsilon\tilde{\Lambda}(\Phi_K) + \tilde{\Omega}(\Phi_K,\Phi_\epsilon),
\end{aligned} \tag{C 1}$$

where the operators on the right-hand side of arise from the expansion of $\tilde{\mathscr{B}}(\tilde{f},\tilde{f},e)$ at $O(K\epsilon)$. The operator $\tilde{\mathscr{L}}$ is defined in (18) and

$$\tilde{\Xi}(\Phi_K) \equiv \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\boldsymbol{u}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}) \left(1-\tfrac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) \mathrm{e}^{-\tilde{u}_2^2}(\Phi_K(\tilde{\boldsymbol{u}}_1') + \Phi_K(\tilde{\boldsymbol{u}}_2')), \qquad \text{(C 2)}$$

$$\tilde{\Omega}(\Phi_K,\Phi_\epsilon) \equiv \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\boldsymbol{u}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}) \mathrm{e}^{-\tilde{u}_2^2} \left(\Phi_K(\tilde{\boldsymbol{u}}_1')\Phi_\epsilon(\tilde{\boldsymbol{u}}_2') + \Phi_K(\tilde{\boldsymbol{u}}_2')\Phi_\epsilon(\tilde{\boldsymbol{u}}_1') - \Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_\epsilon(\tilde{\boldsymbol{u}}_2) - \Phi_K(\tilde{\boldsymbol{u}}_2)\Phi_\epsilon(\tilde{\boldsymbol{u}}_1)\right), \qquad \text{(C 3)}$$

and

$$\tilde{\Lambda}(\Phi_K) = \frac{1}{\pi^{5/2}} \lim_{\epsilon\to 0} \frac{\partial}{\partial\epsilon} \int_{\hat{\boldsymbol{k}}\cdot\boldsymbol{u}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}) \mathrm{e}^{-\tilde{u}_2^2}(\Phi_K(\tilde{\boldsymbol{u}}_1') + \Phi_K(\tilde{\boldsymbol{u}}_2')). \qquad \text{(C 4)}$$

Notice that in the definition of $\tilde{\Omega}$ and $\tilde{\Xi}$ the transformation between the incoming and outgoing velocities is given by the elastic relation. Consider now the operation of the various orders of $\tilde{\mathscr{D}}$ on the left-hand side of (C 1). Clearly, the right-hand side of (13), (14) does not contain any term of order $K\epsilon$. This implies that $\tilde{\mathscr{D}}_{K\epsilon}\log n = \tilde{\mathscr{D}}_{K\epsilon}V_i = 0$. Moreover, the fact that Φ_K does not contribute to Γ or $\tilde{\Gamma}$ (see the text following (28)) implies $\tilde{\Gamma}_K = 0$ (equivalent to $\Gamma_{K\epsilon} = 0$), hence, also $\tilde{\mathscr{D}}_{K\epsilon}\log\Theta = 0$. Next, since the operation of $\tilde{\mathscr{D}}_K$ and $\tilde{\mathscr{D}}_\epsilon$ on the hydrodynamic fields has been computed in the above ((19)–(21) and §3.2) we need only to calculate $\tilde{\mathscr{D}}_K\Phi_\epsilon$ and $\tilde{\mathscr{D}}_\epsilon\Phi_K$. First consider $\tilde{\mathscr{D}}_K\Phi_\epsilon$. Since Φ_ϵ is a function of $\tilde{u}$ alone, which is a function of $\boldsymbol{V}$ and Θ, one obtains by using the chain rule that

$$\tilde{\mathscr{D}}_K\Phi_\epsilon = \epsilon\hat{\Phi}_e'(\tilde{u})\tilde{\mathscr{D}}_K(\tilde{u}^2) = \epsilon\hat{\Phi}_e'(\tilde{u})\tilde{\mathscr{D}}_K\left(\frac{3(\boldsymbol{v}-\boldsymbol{V})^2}{2\Theta}\right)$$
$$= -K\epsilon\hat{\Phi}_e'(\tilde{u})\left(-\tilde{u}_j\frac{\partial\log n}{\partial\tilde{r}_j} + 2\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{1/2}\frac{\partial V_i}{\partial\tilde{r}_j} + (\tilde{u}^2-1)\tilde{u}_j\frac{\partial\log\Theta}{\partial\tilde{r}_j}\right), \qquad \text{(C 5)}$$

where the prime denotes differentiation with respect to $\tilde{u}^2$. In deriving (C 5) we have made use of (20), (21). It remains to evaluate $\tilde{\mathscr{D}}_\epsilon\Phi_K$. Since, as noted above, $\tilde{\mathscr{D}}_\epsilon\log n = \tilde{\mathscr{D}}_\epsilon V_i = 0$, the operation of $\tilde{\mathscr{D}}_\epsilon$ on Φ_K is non-vanishing because of the dependence of the latter on Θ. Notice that Θ appears in Φ_K both explicitly and in $\tilde{u}^2 = 3u^2/2\Theta$, hence by using the relation $\partial\tilde{u}^2/\partial\Theta = -\tilde{u}^2/\Theta$ it follows that

$$\tilde{\mathscr{D}}_\epsilon\Phi_K = \left(-\frac{\partial\Phi_K}{\partial\tilde{u}^2}\frac{\tilde{u}^2}{\Theta} + \frac{\partial\Phi_K}{\partial\Theta}\right)\tilde{\mathscr{D}}_\epsilon\Theta + \frac{\partial\Phi_K}{\partial\left(\partial\log\Theta/\partial r_i\right)}\tilde{\mathscr{D}}_\epsilon\left(\frac{\partial\log\Theta}{\partial r_i}\right). \qquad \text{(C 6)}$$

The action of $\tilde{\mathscr{D}}_\epsilon$ on $\partial\log\Theta/\partial r_i$ can be obtained by using the relation

$$\tilde{\mathscr{D}} = \ell\left(\frac{3}{2\Theta}\right)^{1/2}\left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial r_i}\right).$$

The calculation is facilitated by the fact that the operator in the brackets commutes with $\partial/\partial r_i$, hence one may exchange the order of operations of these two operators.

Next using the result $\tilde{\mathscr{D}}_\epsilon \log \Theta = -\epsilon \frac{2}{3}(2/\pi)^{1/2}$ and $\ell = 1/(\pi n d^2)$, it follows that

$$\tilde{\mathscr{D}}_\epsilon \left(\frac{\partial \log \Theta}{\partial \tilde{r}_i} \right) = -\epsilon \frac{2}{3} \left(\frac{2}{\pi} \right)^{1/2} \frac{\partial}{\partial \tilde{r}_i} \left(\log n + \tfrac{1}{2} \log \Theta \right). \tag{C 7}$$

The result, (C 7), can be used in (C 6), together with the explicit *dimensional* form of Φ_K, to yield

$$\tilde{\mathscr{D}}_\epsilon \Phi_K = K\epsilon \frac{2}{3} \left(\frac{2}{\pi} \right)^{1/2} \left[2 \left(\hat{\Phi}'_v(\tilde{u})\tilde{u}^2 + \tfrac{3}{2} \hat{\Phi}_v(\tilde{u}) \right) \overline{\tilde{u}_i \tilde{u}_j} \left(\frac{3}{2\Theta} \right)^{1/2} \frac{\partial V_i}{\partial \tilde{r}_j} \right.$$
$$\left. + \left(\hat{\Phi}'_c(\tilde{u})\tilde{u}^2 \left(\tilde{u}^2 - \tfrac{5}{2} \right) + \hat{\Phi}_c(\tilde{u})\tilde{u}^2 \right) \tilde{u}_i \frac{\partial \log \Theta}{\partial \tilde{r}_i} - \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \tfrac{5}{2} \right) \tilde{u}_i \frac{\partial \log n}{\partial \tilde{r}_i} \right]. \tag{C 8}$$

All in all, (C 1) assumes the form

$$\tilde{\mathscr{L}}(\Phi_{K\epsilon}) = K\epsilon \left\{ \frac{4}{3} \left(\left(\frac{2}{\pi} \right)^{1/2} (\hat{\Phi}'_v(\tilde{u})\tilde{u}^2 - \hat{\Phi}_v(\tilde{u})(\tilde{u}^2 - 3)) + \tfrac{3}{2}(\hat{\Phi}_e(\tilde{u}) - \hat{\Phi}'_e(\tilde{u})) \right) \right.$$
$$\times \overline{\tilde{u}_i \tilde{u}_j} \left(\frac{3}{2\Theta} \right)^{1/2} \frac{\partial V_i}{\partial \tilde{r}_j} + \left[\hat{\Phi}_e(\tilde{u}) \left(\tilde{u}^2 - \tfrac{5}{2} \right) - \hat{\Phi}'_e(\tilde{u})(\tilde{u}^2 - 1) + \frac{2}{3} \left(\frac{2}{\pi} \right)^{1/2} \right.$$
$$\left. \times \left(\hat{\Phi}'_c(\tilde{u})\tilde{u}^2 \left(\tilde{u}^2 - \frac{5}{2} \right) - \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^4 - 5\tilde{u}^2 + \tfrac{15}{4} \right) \right) \right] \tilde{u}_i \frac{\partial \log \Theta}{\partial \tilde{r}_i}$$
$$\left. + \left(\hat{\Phi}'_e(\tilde{u}) - \frac{2}{3} \left(\frac{2}{\pi} \right)^{1/2} \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \tfrac{5}{2} \right) \right) \tilde{u}_i \frac{\partial \log n}{\partial \tilde{r}_i} \right\} - \epsilon \tilde{\Xi}(\Phi_K)$$
$$- \epsilon \tilde{\Lambda}(\Phi_K) - \tilde{\Omega}(\Phi_K, \Phi_\epsilon)$$
$$\equiv S_{K\epsilon} - \epsilon \tilde{\Xi}(\Phi_K) - \epsilon \tilde{\Lambda}(\Phi_K) - \tilde{\Omega}(\Phi_K, \Phi_\epsilon). \tag{C 9}$$

Note that since the operators in (C 9) are isotropic and so is Φ_ϵ, it follows that $\Phi_{K\epsilon}$ must be of the same tensorial structure as Φ_K, i.e. it includes terms proportional to $\tilde{u}_i$ and $\overline{\tilde{u}_i \tilde{u}_j}$ multiplied by isotropic functions of $\tilde{u}$. This structure of Φ_ϵ combined with the orthogonality conditions imply that this function is orthogonal to any isotropic function. In particular, its contribution to Γ vanishes. The contribution of Φ_ϵ to the heat flux $Q_i^{K\epsilon}$ (see (33)) can be written as a sum of three terms, $Q_{i_1}^{K\epsilon} + Q_{i_2}^{K\epsilon} + Q_{i_3}^{K\epsilon}$. The first term is

$$Q_{i_1}^{K\epsilon} = \frac{n}{2\pi^{3/2}} \left(\frac{2\Theta}{3} \right)^{3/2} \int \mathrm{d}\tilde{\boldsymbol{u}} \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \tfrac{5}{2} \right) \tilde{u}_i \mathrm{e}^{-\tilde{u}^2} S_{K\epsilon}. \tag{C 10}$$

Considering the explicit form of $S_{K\epsilon}$, given in (C 9), it follows by symmetry considerations that its parts, which are proportional to the velocity gradients, vanish upon integration (since the average of $u_i u_j u_k$ vanishes). Hence, the only non-vanishing contributions to $Q_{i_1}^{K\epsilon}$ stem from terms which are proportional to the temperature and density gradients. Next, using the result $\int \mathrm{d}\hat{\tilde{u}} \tilde{u}_i \tilde{u}_j = \frac{4}{3}\pi \tilde{u}^2 \delta_{ij}$, one obtains

$$Q_{i_1}^{K\epsilon} = \alpha_1 \epsilon n \ell \Theta^{1/2} \frac{\partial \Theta}{\partial r_i} + \beta_1 \epsilon \ell \Theta^{3/2} \frac{\partial n}{\partial r_i}, \tag{C 11}$$

where,

$$\alpha_1 \equiv \frac{4}{9}\left(\frac{2}{3\pi}\right)^{1/2}\int_0^\infty \mathrm{d}\tilde{u}\tilde{u}^4\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\tfrac{5}{2}\right)\mathrm{e}^{-\tilde{u}^2} \times\left[\hat{\Phi}_e(\tilde{u})\left(\tilde{u}^2-\tfrac{5}{2}\right)-\hat{\Phi}'_e(\tilde{u})(\tilde{u}^2-1)+\frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2} \times\left(\hat{\Phi}'_c(\tilde{u})\tilde{u}^2\left(\tilde{u}^2-\tfrac{5}{2}\right)-\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^4-5\tilde{u}^2+\tfrac{15}{4}\right)\right)\right], \tag{C 12}$$

and,

$$\beta_1 \equiv \frac{4}{9}\left(\frac{2}{3\pi}\right)^{1/2}\int_0^\infty \mathrm{d}\tilde{u}\tilde{u}^4\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\tfrac{5}{2}\right)\mathrm{e}^{-\tilde{u}^2}\left(\hat{\Phi}'_e(\tilde{u})-\frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\tfrac{5}{2}\right)\right). \tag{C 13}$$

Both of the above integrals have been evaluated numerically. The result is $\alpha_1 \approx -0.3619$ and $\beta_1 \approx -0.2110$. The second term contributing to $Q_i^{K\epsilon}$ is

$$Q_{i_2}^{K\epsilon} = -\frac{\epsilon n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{3/2}\int \mathrm{d}\tilde{\boldsymbol{u}}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\tfrac{5}{2}\right)\tilde{u}_i\mathrm{e}^{-\tilde{u}^2}(\tilde{\Xi}+\tilde{\Lambda}). \tag{C 14}$$

The reason that we consider $\tilde{\Xi}$ and $\tilde{\Lambda}$ together is that this way one obtains a cancelation of terms. Substituting (C 2) and (C 4) in (C 14), one obtains

$$\begin{aligned} Q_{i_2}^{K\epsilon} = &-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2}\int_{\hat{\boldsymbol{k}}\cdot\boldsymbol{u}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left(1-\tfrac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right)\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\ &\times(\Phi_K(\tilde{\boldsymbol{u}}'_1)+\Phi_K(\tilde{\boldsymbol{u}}'_2))\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\tfrac{5}{2}\right)\tilde{u}_{1i} \\ &-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\boldsymbol{u}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\ &\times(\Phi_K(\tilde{\mathbf{u}}'_1)+\Phi_K(\tilde{\boldsymbol{u}}'_2))\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\tfrac{5}{2}\right)\tilde{u}_{1i}. \end{aligned} \tag{C 15}$$

Recall that in the first integral the elastic velocity transformation is employed while in the second integral one has to consider the inelastic transformation to first order in ϵ. Hence, in the first integral one can transform to primed integration variables by using $\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2 = \mathrm{d}\tilde{\boldsymbol{u}}'_1\mathrm{d}\tilde{\boldsymbol{u}}'_2$, $\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12} = -\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12}$ and $\tilde{u}_1^2+\tilde{u}_2^2 = \tilde{u}_1'^2+\tilde{u}_2'^2$, and in the second integral one must use $\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2 = (1-\frac{1}{2}\epsilon)\mathrm{d}\tilde{\boldsymbol{u}}'_1\mathrm{d}\tilde{\boldsymbol{u}}'_2$, $\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12} = -(1-\frac{1}{2}\epsilon)\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12}$ and $\tilde{u}_1^2+\tilde{u}_2^2 = \tilde{u}_1'^2+\tilde{u}_2'^2-\frac{1}{2}\epsilon(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12})^2$. Next, upon taking the derivative with respect to ϵ at $\epsilon = 0$ (Cf. (C 15)) one obtains

$$\begin{aligned} Q_{i_2}^{K\epsilon} = &-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\boldsymbol{u}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\ &\times(\Phi_K(\tilde{\boldsymbol{u}}_1)+\Phi_K(\tilde{\boldsymbol{u}}_2))\hat{\Phi}_c(\tilde{u}'_1)\left(\tilde{u}_1'^2-\tfrac{5}{2}\right)\tilde{u}'_{1i}. \end{aligned} \tag{C 16}$$

Considering the form of Φ_K (cf. (23)), it follows by symmetry considerations that the part of Φ_K which is proportional to $\overline{\tilde{u}_i\tilde{u}_j}$ does not contribute to the integral in (C 16). The integral is further simplified by renaming $\tilde{\boldsymbol{u}}'_1$ as $\tilde{\boldsymbol{u}}$, multiplying the integrand by

$\delta(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1 + q(\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}})$ (where $q \equiv \frac{1}{2}(1+e)$) and integrating over $\tilde{\boldsymbol{u}}$. One obtains

$$Q_{i_2}^{K\epsilon} = -\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\boldsymbol{u} \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \times (\Phi_K(\tilde{\boldsymbol{u}}_1) + \Phi_K(\tilde{\boldsymbol{u}}_2))\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \tfrac{5}{2}\right)\tilde{u}_i I_\delta, \tag{C 17}$$

where, $I_\delta \equiv \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})\delta(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1 + q(\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}})$. The integral in (C 17) is then split into two parts. The first part is (using (23) and defining $\tilde{\boldsymbol{s}} \equiv \tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1$)

$$(I) = -\frac{\epsilon n K}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2} \frac{\partial \log\Theta}{\partial \tilde{r}_j} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\boldsymbol{u} \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2} \times \hat{\Phi}_c(|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u})\left((\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}})^2 - \tfrac{5}{2}\right)\left(\tilde{u}^2 - \tfrac{5}{2}\right)\tilde{u}_i(\tilde{u}_j - \tilde{s}_j)I_\delta. \tag{C 18}$$

First, one performs an integration over $\hat{\tilde{\boldsymbol{u}}}_2$ (i.e. the orientations of $\tilde{\boldsymbol{u}}_2$). To this end one needs (D 13), which shows that for any smooth function, F, the following holds:

$$\int_0^\pi \mathrm{d}\theta_2' \sin\theta_2' F(\cos\theta_2') I_\delta = \frac{1}{q^2 s u_2} F\left(\frac{\hat{\boldsymbol{s}}\cdot\boldsymbol{u}}{u_2} + \frac{1-q}{q}\frac{s}{u_2}\right) H\left(u_2 - \left|\frac{1-q}{q}s + \hat{\boldsymbol{s}}\cdot\boldsymbol{u}\right|\right).$$

It follows that

$$\int \mathrm{d}\hat{\tilde{\boldsymbol{u}}}_2 I_\delta = \frac{2\pi}{q^2 \tilde{s}\tilde{u}_2} H\left(\tilde{u}_2 - \left|\frac{1-q}{q}\tilde{s} + \hat{\tilde{\boldsymbol{s}}}\cdot\tilde{\boldsymbol{u}}\right|\right),$$

where H denotes the Heaviside function. Hence (C 18) assumes the form

$$(I) = -\frac{\epsilon n K}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2} \frac{\partial \log\Theta}{\partial \tilde{r}_j} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{2\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}} \int_{|\frac{1-q}{q}\tilde{s}+\hat{\tilde{\boldsymbol{s}}}\cdot\tilde{\boldsymbol{u}}|}^\infty \mathrm{d}\tilde{u}_2 \frac{\tilde{u}_2}{\tilde{s}} \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2} \times \hat{\Phi}_c(|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u})\left((\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}})^2 - \tfrac{5}{2}\right)\left(\tilde{u}^2 - \tfrac{5}{2}\right)\tilde{u}_i(\tilde{u}_j - \tilde{s}_j), \tag{C 19}$$

Clearly, the integral in (C 19) is proportional to the second-order isotropic tensor, δ_{ij}. One may, therefore, choose specific directions in the integral, e.g. $i = j = 1$, and replace $\partial\log\Theta/\partial\tilde{r}_j$ by $\partial\log\Theta/\partial\tilde{r}_i$. Next one performs an integration over the orientations of $\tilde{\boldsymbol{s}}$ and $\tilde{\boldsymbol{u}}$ keeping the angle between them, θ', fixed. This integration is performed by first transforming $\tilde{\boldsymbol{s}}$ to spherical coordinates, choosing the z-axis to coincide with $\tilde{\boldsymbol{u}}$ and integrating over the azimuthal angle of $\tilde{\boldsymbol{s}}$ in the latter frame of reference. An integration over all orientations of $\tilde{\boldsymbol{u}}$ is performed next. It follows that: $\int_{\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{s}}=\tilde{u}\tilde{s}\cos\theta'} \mathrm{d}\hat{\tilde{\boldsymbol{u}}}\,\mathrm{d}\hat{\tilde{\boldsymbol{s}}}\tilde{u}_x(\tilde{u}_x - \tilde{s}_x) = \frac{8}{3}\pi^2\tilde{u}(\tilde{u} - \tilde{s}\cos\theta')$, where use has been made of the transformations given in (A 3) and (A 4). Substituting this result in (C 19), performing an obvious shift in the definition of $\tilde{u}_2$ and substituting $\mathrm{d}y \equiv \mathrm{d}(\cos\theta')$, one obtains

$$(I) = -\frac{\epsilon n K}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2} \frac{\partial \log\Theta}{\partial \tilde{r}_i} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{16\pi^3}{3q^2}\int_0^\infty \mathrm{d}\tilde{s}\int_0^\infty \mathrm{d}\tilde{u}\int_0^\infty \mathrm{d}\tilde{u}_2 \int_{-1}^1 \mathrm{d}y\,\tilde{s}\tilde{u}_2\tilde{u}^3(\tilde{u} - \tilde{s}y) \times \hat{\Phi}_c\left((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2}\right)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \tfrac{5}{2}\right)\left(\tilde{u}^2 - \tfrac{5}{2}\right) \times \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\mathrm{e}^{-(\tilde{u}_2^2 + (\frac{1-q}{q}\tilde{s} + \tilde{u}y)^2)}. \tag{C 20}$$

Next, taking the derivative with respect to ϵ at $\epsilon = 0$ and carrying out the integration over $\tilde{u}_2$, one obtains $(I) = \alpha_2 \epsilon n \ell \Theta^{1/2}\partial\Theta/\partial r_i$ where

$$\alpha_2 = -\frac{4\sqrt{2}}{9\sqrt{3\pi}}\int_0^\infty \mathrm{d}\tilde{s}\int_0^\infty \mathrm{d}\tilde{u}\int_{-1}^1 \mathrm{d}y\,\tilde{s}\tilde{u}^3(\tilde{u} - \tilde{s}y)(1 - \tilde{s}\tilde{u}y) \times \hat{\Phi}_c\left((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2}\right)\hat{\Phi}_c(\tilde{u})(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \tfrac{5}{2})\left(\tilde{u}^2 - \tfrac{5}{2}\right) \times \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\mathrm{e}^{-\tilde{u}^2 y^2}. \tag{C 21}$$

The triple integral in (C 21) has been evaluated numerically. The result is $\alpha_2 \approx -0.0282$. The second part of (C 17) reads

$$(II) = -\frac{\epsilon n K}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2}\frac{\partial \log \Theta}{\partial \tilde{r}_j}\lim_{\epsilon\to 0}\frac{\partial}{\partial \epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\boldsymbol{u}\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}$$
$$\times\, \hat{\Phi}_c(\tilde{u}_2)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}_2^2-\tfrac{5}{2}\right)\left(\tilde{u}^2-\tfrac{5}{2}\right)\tilde{u}_i\tilde{u}_{2j}I_\delta. \qquad \text{(C 22)}$$

Using similar consideration as in the previous case together with the result (D 13), cited above,

$$\int \mathrm{d}\hat{\tilde{\boldsymbol{u}}}_2\tilde{u}_{2x}I_\delta = \frac{2\pi\tilde{s}_x}{q^2\tilde{s}^2\tilde{u}_2}\left(\frac{1-q}{q}\tilde{s}+\hat{\tilde{\boldsymbol{s}}}\cdot\tilde{\boldsymbol{u}}\right)H\left(\tilde{u}_2-|\frac{1-q}{q}\tilde{s}+\hat{\tilde{\boldsymbol{s}}}\cdot\tilde{\boldsymbol{u}}|\right),$$

one obtains

$$(II) = -\frac{\epsilon n K}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2}\frac{\partial \log \Theta}{\partial \tilde{r}_i}\lim_{\epsilon\to 0}\frac{\partial}{\partial \epsilon}\frac{2\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\int_{|\frac{1-q}{q}\tilde{s}+\hat{\tilde{\boldsymbol{s}}}\cdot\tilde{\boldsymbol{u}}|}^{\infty}\mathrm{d}\tilde{u}_2\frac{\tilde{u}_2}{\tilde{s}^2}\left(\frac{1-q}{q}\tilde{s}+\hat{\tilde{\boldsymbol{s}}}\cdot\tilde{\boldsymbol{u}}\right)$$
$$\times\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\hat{\Phi}_c(\tilde{u}_2)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}_2^2-\tfrac{5}{2}\right)\left(\tilde{u}^2-\tfrac{5}{2}\right)\tilde{u}_x\tilde{s}_x. \qquad \text{(C 23)}$$

Next using the result:

$$\int_{\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{s}}=\tilde{u}\tilde{s}\cos\theta'}\mathrm{d}\hat{\tilde{\boldsymbol{u}}}\mathrm{d}\hat{\tilde{\boldsymbol{s}}}\tilde{u}_x\tilde{s}_x = \frac{8\pi^2}{3}\tilde{u}\tilde{s}\cos\theta',$$

followed by an appropriate shift in the integration variable $\tilde{u}_2$ and the substitution $\mathrm{d}y = \mathrm{d}(\cos\theta')$, one obtains

$$(II) = -\frac{\epsilon n K}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2}\frac{\partial \log \Theta}{\partial \tilde{r}_i}\lim_{\epsilon\to 0}\frac{\partial}{\partial \epsilon}\frac{16\pi^3}{3q^2}\int_0^\infty \mathrm{d}\tilde{s}\int_0^\infty \mathrm{d}\tilde{u}\int_0^\infty d\tilde{u}_2\int_{-1}^{1}\mathrm{d}y\tilde{s}\tilde{u}_2\tilde{u}^3y$$
$$\times\left(\frac{1-q}{q}\tilde{s}+\tilde{u}y\right)\times\hat{\Phi}_c\left(\left(\tilde{u}_2^2+\left(\frac{1-q}{q}\tilde{s}+\tilde{u}y\right)^2\right)^{1/2}\right)\hat{\Phi}_c(\tilde{u})$$
$$\times\left(\tilde{u}_2^2+\left(\frac{1-q}{q}\tilde{s}+\tilde{u}y\right)^2-\frac{5}{2}\right)\left(\tilde{u}^2-\tfrac{5}{2}\right)\times\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\mathrm{e}^{-(\tilde{u}_2^2+(\frac{1-q}{q}\tilde{s}+\tilde{u}y)^2)}. \qquad \text{(C 24)}$$

Taking the derivative with respect to ϵ at $\epsilon = 0$ and carrying out the integration over $\tilde{s}$ yields $(II) = \alpha_3\epsilon n\ell\Theta^{1/2}\partial\Theta/\partial r_i$, where

$$\alpha_3 = -\frac{4\sqrt{2}}{9\sqrt{3}\pi}\int_0^\infty \mathrm{d}\tilde{u}\int_0^\infty \mathrm{d}\tilde{u}_2\int_{-1}^{1}\mathrm{d}y\tilde{u}_2\tilde{u}^3y$$
$$\times\left\{\left[\tilde{u}y+\frac{\sqrt{\pi}}{2}(1+2\tilde{u}^2y^2)\mathrm{e}^{\tilde{u}^2y^2}(1+\mathrm{erf}(\tilde{u}y))\right]\right.$$
$$\times\left[\frac{1}{2}-\tilde{u}^2y^2\left(1-\frac{\hat{\Phi}_c'\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)}{\hat{\Phi}_c\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)}-\frac{1}{\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}}\right)\right]$$
$$\left.+\,\tilde{u}y+\pi^{1/2}\tilde{u}^2y^2\mathrm{e}^{\tilde{u}^2y^2}(1+\mathrm{erf}(\tilde{u}y))\right\}\left(\tilde{u}_2^2+\tilde{u}^2y^2-\tfrac{5}{2}\right)\left(\tilde{u}^2-\tfrac{5}{2}\right)$$
$$\times\,\hat{\Phi}_c\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)\hat{\Phi}_c(\tilde{u})\mathrm{e}^{-\tilde{u}^2}\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}. \qquad \text{(C 25)}$$

The above triple integral has been evaluated numerically. The result is $\alpha_3 \approx 0.2849$. The third contribution to $Q_i^{K\epsilon}$ is

$$Q_{i_3}^{K\epsilon} = -\frac{\epsilon n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{3/2}\int d\tilde{\boldsymbol{u}}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\tfrac{5}{2}\right)\tilde{u}_i e^{-\tilde{u}^2}\tilde{\Omega}, \tag{C 26}$$

where $\tilde{\Omega}$ is given in (C 3). Substituting (C 3) in (C 26), one obtains

$$\begin{aligned} Q_{i_3}^{K\epsilon} = &-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2}\int_{\hat{\boldsymbol{k}}\cdot\boldsymbol{u}_{12}>0} d\tilde{\boldsymbol{u}}_1 d\tilde{\boldsymbol{u}}_2 d\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\tfrac{5}{2}\right)\tilde{u}_{1i} \\ &\times\left(\Phi_K(\tilde{\boldsymbol{u}}_1')\Phi_\epsilon(\tilde{\boldsymbol{u}}_2')+\Phi_K(\tilde{\boldsymbol{u}}_2')\Phi_\epsilon(\tilde{\boldsymbol{u}}_1')-\Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_\epsilon(\tilde{\boldsymbol{u}}_2)-\Phi_K(\tilde{\boldsymbol{u}}_2)\Phi_\epsilon(\tilde{\boldsymbol{u}}_1)\right). \end{aligned} \tag{C 27}$$

Notice that in (C 27), the velocity transformation corresponds to the elastic limit. The part of the integral involving $(\Phi_K(\tilde{\boldsymbol{u}}_1')\Phi_\epsilon(\tilde{\boldsymbol{u}}_2')+\Phi_K(\tilde{\boldsymbol{u}}_2')\Phi_\epsilon(\tilde{\boldsymbol{u}}_1'))$ can be transformed to an integration over the primed variables by using the elastic transformation, cf. the text following (C 15), and an exchange of the primed and unprimed variables. This part of the integral is further simplified by renaming $\tilde{\boldsymbol{u}}_1'$ as $\tilde{\boldsymbol{u}}$, multiplying the integrand by $\delta(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1'+(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}})$ and integrating over $\tilde{\boldsymbol{u}}$. One obtains

$$\begin{aligned} Q_{i_3}^{K\epsilon} = &-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2}\int d\tilde{\boldsymbol{u}}_1 d\tilde{\boldsymbol{u}}_2 d\tilde{\boldsymbol{u}} e^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\ &\times\left(\Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_\epsilon(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_2)\Phi_\epsilon(\tilde{\boldsymbol{u}}_1)\right)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\tfrac{5}{2}\right)\tilde{u}_i I_\delta^{(0)} \\ &+\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2}\int_{\hat{\boldsymbol{k}}\cdot\boldsymbol{u}_{12}>0} d\tilde{\boldsymbol{u}}_1 d\tilde{\boldsymbol{u}}_2 d\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})e^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\ &\times\left(\Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_\epsilon(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_2)\Phi_\epsilon(\tilde{\boldsymbol{u}}_1)\right)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\tfrac{5}{2}\right)\tilde{u}_{1i}, \end{aligned} \tag{C 28}$$

where $I_\delta^{(0)} \equiv I_\delta(q=1)$. The term $Q_{i_3}^{K\epsilon}$ is split into four parts. The first equals (after employing the explicit forms of Φ_K and Φ_ϵ, and using $\tilde{\boldsymbol{s}} \equiv \tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1$)

$$\begin{aligned} (I) = &-\frac{\epsilon n K}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2}\frac{\log\Theta}{\partial\tilde{r}_j}\int d\tilde{\boldsymbol{s}} d\tilde{\boldsymbol{u}}_2 d\tilde{\boldsymbol{u}} e^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2} \\ &\times\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_c(\tilde{u})\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tfrac{5}{2}\right)\left(\tilde{u}^2-\tfrac{5}{2}\right)\tilde{u}_i(\tilde{u}_j-\tilde{s}_j)I_\delta^{(0)}. \end{aligned} \tag{C 29}$$

Notice that except for the extra term $\hat{\Phi}_e(\tilde{u}_2)$ and the above definition of $I_\delta^{(0)}$, the integrand in (C 29) is similar to that in (C 18). Following a similar derivation as performed following (C 18), one obtains $(I) = \alpha_4\epsilon n\ell\Theta^{1/2}\partial\Theta/\partial r_i$ where

$$\begin{aligned} \alpha_4 = &-\frac{16\sqrt{2}}{9\sqrt{3\pi}}\int_0^\infty d\tilde{s}\int_0^\infty d\tilde{u}\int_0^\infty d\tilde{u}_2\int_{-1}^1 dy\tilde{s}\tilde{u}_2\tilde{u}^3(\tilde{u}-\tilde{s}y)\hat{\Phi}_c(\tilde{u}) \\ &\times\hat{\Phi}_c\left((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\right)\hat{\Phi}_e\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)\left(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2-\tfrac{5}{2}\right)\left(\tilde{u}^2-\tfrac{5}{2}\right) \\ &\times e^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}e^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}. \end{aligned} \tag{C 30}$$

The integral in (C 30) has been evaluated numerically. The result is $\alpha_4 \approx -0.0016$.

The second term reads

$$\begin{aligned} (II) = &-\frac{\epsilon n K}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{3/2}\frac{\partial\log\Theta}{\partial\tilde{r}_j}\int d\tilde{\boldsymbol{s}} d\tilde{\boldsymbol{u}}_2 d\tilde{\boldsymbol{u}} e^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2} \\ &\times\hat{\Phi}_c(\tilde{u}_2)\hat{\Phi}_e(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}_2^2-\tfrac{5}{2}\right)\left(\tilde{u}^2-\tfrac{5}{2}\right)\tilde{u}_i\tilde{u}_{2j}I_\delta^{(0)}. \end{aligned} \tag{C 31}$$

Notice that except for the extra term $\hat{\Phi}_e(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)$ and the above definition of $I_\delta^{(0)}$, the integrand in (C 31) is similar to the one in (C 22). Therefore a derivation similar to that presented following (C 22) results in $(II) = \alpha_5 \epsilon n \ell \Theta^{1/2} \partial\Theta/\partial r_i$, where

$$\alpha_5 = -\frac{16\sqrt{2}}{9\sqrt{3}\pi}\int_0^\infty d\tilde{s}\int_0^\infty d\tilde{u}\int_0^\infty d\tilde{u}_2\int_{-1}^1 dy\,\tilde{s}\tilde{u}_2\tilde{u}^4 y^2\hat{\Phi}_c(\tilde{u})$$
$$\times\,\hat{\Phi}_c\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)\hat{\Phi}_e\left((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\right)\left(\tilde{u}_2^2+\tilde{u}^2y^2-\tfrac{5}{2}\right)\left(\tilde{u}^2-\tfrac{5}{2}\right)$$
$$\times\,e^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}e^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}. \qquad (C\,32)$$

The integral in (C 32) has been evaluated numerically. The result is $\alpha_5 \approx -0.0016$. The computation of the third and fourth parts is much simpler since the corresponding integrands do not include a mixture of precollisional and postcollisional velocities. This fact renders the integrations over $\hat{\boldsymbol{k}}$ trivial. Hence, using $\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} d\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}) = \pi u_{12}$, the third part reads, after substituting the forms of Φ_K and Φ_ϵ (cf. (23) and the text following (30))

$$(III) = \frac{\epsilon n K}{2\pi^3}\left(\frac{2\Theta}{3}\right)^{3/2}\frac{\partial\Theta}{\partial r_j}\int d\tilde{\boldsymbol{u}}_1 d\tilde{\boldsymbol{u}}_2\tilde{u}_{12}e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_c^2(\tilde{u}_1)\left(\tilde{u}_1^2-\tfrac{5}{2}\right)^2\tilde{u}_{1i}\tilde{u}_{1j}. \qquad (C\,33)$$

Next, using $\int_{\tilde{\boldsymbol{u}}_1\cdot\tilde{\boldsymbol{u}}_2=\tilde{u}_1\tilde{u}_2\cos\theta'} d\hat{\tilde{\boldsymbol{u}}}_1 d\hat{\tilde{\boldsymbol{u}}}_2\tilde{u}_{1x}^2 = \frac{8}{3}\pi^2\tilde{u}_1^2$, it follows that $(III) = \alpha_6\epsilon n\ell\Theta^{1/2}\partial\Theta/\partial r_i$ where

$$\alpha_6 = \frac{8\sqrt{2}}{9\sqrt{3}\pi}\int_0^\infty d\tilde{u}_1\int_0^\infty d\tilde{u}_2\tilde{u}_1^4\tilde{u}_2^2 R_0(\tilde{u}_1,\tilde{u}_2)e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\times\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_c^2(\tilde{u}_1)\left(\tilde{u}_1^2-\tfrac{5}{2}\right)^2. \qquad (C\,34)$$

The function R_n is defined as

$$R_n(\tilde{u}_1,\tilde{u}_2) \equiv \int_0^\pi d\theta'\sin\theta' P_n(\cos\theta')(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta'+\tilde{u}_2^2)^{1/2}, \qquad (C\,35)$$

where $P_n(x)$ is the nth-order Legendre polynomial. Substituting $n = 0$, one obtains (Pekeris 1955) (for $\tilde{u}_1 > \tilde{u}_2$): $R_0(\tilde{u}_1,\tilde{u}_2) = 2\tilde{u}_2^2/3\tilde{u}_1 + 2\tilde{u}_1$; the value of R for $\tilde{u}_2 > \tilde{u}_1$ is obtained by exchanging the order of its arguments. The double integral in (C 34) has been carried out numerically. The result is $\alpha_6 \approx 0.0018$. The fourth part is calculated in a similar manner. After integrating over $\hat{\boldsymbol{k}}$, employing the forms of Φ_K and Φ_ϵ and using $\int_{\tilde{\boldsymbol{u}}_1\cdot\tilde{\boldsymbol{u}}_2=\tilde{u}_1\tilde{u}_2\cos\theta'} d\hat{\tilde{\boldsymbol{u}}}_1 d\hat{\tilde{\boldsymbol{u}}}_2\tilde{u}_{1x}\tilde{u}_{2x} = \frac{8}{3}\pi^2\tilde{u}_1\tilde{u}_2\cos\theta'$ one obtains $(IV) = \alpha_7\epsilon n\ell\Theta^{1/2}\partial\Theta/\partial r_i$ where

$$\alpha_7 = \frac{8\sqrt{2}}{9\sqrt{3}\pi}\int_0^\infty d\tilde{u}_1\int_0^\infty d\tilde{u}_2\tilde{u}_1^3\tilde{u}_2^3 R_1(\tilde{u}_1,\tilde{u}_2)e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}$$
$$\times\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\tfrac{5}{2}\right)\left(\tilde{u}_2^2-\tfrac{5}{2}\right), \qquad (C\,36)$$

with R_1 given by Pekeris (1955) (for $\tilde{u}_1 > \tilde{u}_2$):

$$R_1(\tilde{u}_1,\tilde{u}_2) = \frac{2\tilde{u}_2^3}{15\tilde{u}_1^2} - \frac{2}{3}\tilde{u}_2. \qquad (C\,37)$$

The double integral in (C 36) has been evaluated numerically. The result is $\alpha_7 \approx -0.0006$. Summing all the contributions to $Q_i^{K\epsilon}$ one obtains

$$Q_i^{K\epsilon} = -\kappa_1\frac{\partial\Theta}{\partial r_i} - \lambda_1\frac{\partial n}{\partial r_i}, \qquad (C\,38)$$

where $\kappa_1 \approx 0.1072\epsilon n\ell\Theta^{1/2}$ and $\lambda_1 \approx 0.2110\epsilon\ell\Theta^{3/2}$.

Next, consider the $K\epsilon$ order of the stress tensor. Following (36) and (C 9), this contribution to the stress tensor can be written as the sum of three terms: $P_{ij}^{K\epsilon} = P_{ij_1}^{K\epsilon} + P_{ij_2}^{K\epsilon} + P_{ij_3}^{K\epsilon}$. The first term reads

$$P_{ij_1}^{K\epsilon} = \frac{2n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}} \hat{\Phi}_v(\tilde{u}) \overline{\tilde{u}_i \tilde{u}_j} \mathrm{e}^{-\tilde{u}^2} S_{K\epsilon}. \tag{C 39}$$

Symmetry considerations applied to the explicit form of $S_{K\epsilon}$ (cf. (C 9)) imply that only the term proportional to velocity gradients contributes to the integral in (C 39). Hence, upon substituting (C 9) in (C 39), one obtains

$$P_{ij_1}^{K\epsilon} = \frac{4\epsilon n K}{3\pi^{3/2}} \left(\frac{2\Theta}{3}\right)^{1/2} \overline{\frac{\partial V_k}{\partial r_\ell}} \int \mathrm{d}\tilde{\boldsymbol{u}} \hat{\Phi}_v(\tilde{u}) \overline{\tilde{u}_i \tilde{u}_j} \tilde{u}_k \tilde{u}_\ell \mathrm{e}^{-\tilde{u}^2} \\ \times \left(\left(\frac{2}{\pi}\right)^{1/2} (\hat{\Phi}'_v(\tilde{u})\tilde{u}^2 - \hat{\Phi}_v(\tilde{u})(\tilde{u}^2 - 3)) + \tfrac{3}{2}(\hat{\Phi}_e(\tilde{u}) - \hat{\Phi}'_e(\tilde{u})) \right), \tag{C 40}$$

where use has been made of the tensorial identity: $\overline{a_i a_j} T_{ij} = a_i a_j \overline{T_{ij}}$. Clearly, the above integral is proportional to an isotropic fourth-order tensor, which is symmetric with respect to the exchange of pairs (i, j) and (k, ℓ): $\delta_{ij}\delta_{k\ell} + b(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$ The vanishing of the trace of $\overline{\partial V_k/\partial r_\ell}$ implies that the term $\delta_{ij}\delta_{k\ell}$ does not contribute to $P_{ij_1}^{K\epsilon}$. Each of the other two Kronecker δ yields the same contribution. Hence, in order to calculate the expression in (C 40) one may choose e.g. $i = k = 1$ and $j = \ell = 2$, replace $\partial \log \Theta / \partial \tilde{r}_j$ by $\partial \log \Theta / \partial \tilde{r}_i$, and multiply the result by 2. Next, using $\int \mathrm{d}\hat{\tilde{\boldsymbol{u}}} \tilde{u}_x^2 \tilde{u}_y^2 = \frac{4}{15}\pi \tilde{u}^4$, it follows that $P_{ij_1}^{K\epsilon} = \zeta_1 \epsilon n \ell \Theta^{1/2} \overline{\partial V_i / \partial r_j}$, where

$$\zeta_1 = \frac{32}{45} \left(\frac{2}{3\pi}\right)^{1/2} \int_0^\infty \mathrm{d}\tilde{u} \tilde{u}^6 \hat{\Phi}_v(\tilde{u}) \mathrm{e}^{-\tilde{u}^2} \\ \left(\left(\frac{2}{\pi}\right)^{1/2} (\hat{\Phi}'_v(\tilde{u})\tilde{u}^2 - \hat{\Phi}_v(\tilde{u})(\tilde{u}^2 - 3)) + \tfrac{3}{2}(\hat{\Phi}_e(\tilde{u}) - \hat{\Phi}'_e(\tilde{u})) \right), \tag{C 41}$$

Carrying out the integration numerically, one obtains $\zeta_1 \approx -0.0935$.

The second term reads

$$P_{ij_2}^{K\epsilon} = -\frac{2n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}} \hat{\Phi}_v(\tilde{u}) \overline{\tilde{u}_i \tilde{u}_j} \mathrm{e}^{-\tilde{u}^2} (\tilde{\Xi} + \tilde{\Lambda}). \tag{C 42}$$

Upon substituting (C 2) and (C 4) in (C 42), and taking of the derivative with respect to ϵ at $\epsilon = 0$ (cf. the derivation of $Q_{i_2}^{K\epsilon}$) one obtains

$$P_{ij_2}^{K\epsilon} = -\frac{2\epsilon n\Theta}{3\pi^4} \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \int_{\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12} > 0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12}) \mathrm{e}^{-(\tilde{u}_1^2 + \tilde{u}_2^2)} \\ \times (\Phi_K(\tilde{\boldsymbol{u}}_1) + \Phi_K(\tilde{\boldsymbol{u}}_2)) \hat{\Phi}_v(\tilde{u}'_1) \overline{\tilde{u}'_{1i} \tilde{u}'_{1j}}, \tag{C 43}$$

where primes now denote postcollisional velocities. Considering the form of Φ_K, cf. (23), symmetry considerations imply that only the part which is proportional to velocity gradients contributes to the integral in (C 43). Further simplification is achieved by renaming $\tilde{\boldsymbol{u}}'_1$ as $\tilde{\boldsymbol{u}}$, multiplying the integrand by $\delta(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1 + q(\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}})$ and integrating over $\tilde{\boldsymbol{u}}$. One obtains

$$P_{ij_2}^{K\epsilon} = -\frac{2\epsilon n\Theta}{3\pi^4} \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}} \mathrm{e}^{-(\tilde{u}_1^2 + \tilde{u}_2^2)} (\Phi_K(\tilde{\boldsymbol{u}}_1) + \Phi_K(\tilde{\boldsymbol{u}}_2)) \hat{\Phi}_v(\tilde{u}) \overline{\tilde{u}_i \tilde{u}_j} I_\delta, \tag{C 44}$$

where I_δ and q are defined in the above. Next, the expression on the right-hand side of (C 44) is split into two parts, the first being (after employing the explicit form of Φ_K and using $\tilde{\boldsymbol{s}} \equiv \tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1$)

$$(I) = -\frac{4\epsilon K n\Theta}{3\pi^4}\left(\frac{3}{2\Theta}\right)^{1/2}\overline{\frac{\partial V_k}{\partial \tilde{r}_\ell}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2} \times \hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)(\tilde{u}_\ell-\tilde{s}_\ell)I_\delta. \tag{C 45}$$

An integration over all the orientations of $\tilde{\boldsymbol{u}}_2$ (using (D 13)) followed by the application of tensorial arguments (which imply that one may choose specific directions $i = k = 1$ and $j = \ell = 2$, replace $\overline{\partial V_k/\partial r_\ell}$ by $\overline{\partial V_i/\partial r_j}$ and multiply the result by 2) yields

$$(I) = -\frac{4\epsilon K n\Theta}{3\pi^4}\left(\frac{3}{2\Theta}\right)^{1/2}\overline{\frac{\partial V_i}{\partial \tilde{r}_j}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{4\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\int_{|\frac{1-q}{q}\tilde{s}+\hat{\tilde{\boldsymbol{s}}}\cdot\tilde{\boldsymbol{u}}|}^{\infty}\mathrm{d}\tilde{u}_2\tilde{u}_2\frac{1}{\tilde{s}}\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2} \times \hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u})\tilde{u}_x\tilde{u}_y(\tilde{u}_x-\tilde{s}_x)(\tilde{u}_y-\tilde{s}_y). \tag{C 46}$$

Using: $\int_{\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{s}}=\tilde{u}\tilde{s}\cos\theta'}\mathrm{d}\hat{\tilde{\boldsymbol{u}}}\mathrm{d}\hat{\tilde{\boldsymbol{s}}}\tilde{u}_x\tilde{u}_y(\tilde{u}_x-\tilde{s}_x)(\tilde{u}_y-\tilde{s}_y) = \frac{8}{15}\pi^2\tilde{u}^2\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\frac{1}{2}\tilde{s}^2(3\cos^2\theta'-1)\right)$ together with an appropriate shift in $\tilde{u}_2$ and the substitution $\mathrm{d}y \equiv \mathrm{d}(\cos\theta')$, then taking the derivative with respect to ϵ at $\epsilon = 0$ and integrating over $\tilde{u}_2$, one obtains $(I) = \zeta_2\epsilon n\ell\Theta^{1/2}\overline{\partial V_i/\partial r_j}$, where

$$\zeta_2 = -\frac{32\sqrt{3}}{45\sqrt{2\pi}}\int_0^\infty \mathrm{d}\tilde{s}\int_0^\infty \mathrm{d}\tilde{u}\int_{-1}^{1}\mathrm{d}y\tilde{s}\tilde{u}^4\left(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tfrac{1}{2}\tilde{s}^2(3y^2-1)\right) \times (1-\tilde{s}\tilde{u}y)\hat{\Phi}_v(\sqrt{\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2})\hat{\Phi}_v(\tilde{u})\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\mathrm{e}^{-\tilde{u}^2y^2} \tag{C 47}$$

The integral has been evaluated numerically. The result is $\zeta_2 \approx -0.1349$. The second part of $P_{ij_2}^{K\epsilon}$ is

$$(II) = -\frac{4\epsilon K n\Theta}{3\pi^4}\left(\frac{3}{2\Theta}\right)^{1/2}\overline{\frac{\partial V_k}{\partial \tilde{r}_\ell}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\,\mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\hat{\Phi}_v(\tilde{u}_2)\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_{2k}\tilde{u}_{2\ell}I_\delta. \tag{C 48}$$

As before we choose specific directions such as $i = k = 1$ and $j = \ell = 2$, replace $\overline{\partial V_k/\partial r_\ell}$ by $\overline{\partial V_i/\partial r_j}$ and multiply the result by 2. Next, upon performing the integration over the orientations of $\tilde{\boldsymbol{u}}_2$ and, using (D 13):

$$\int \mathrm{d}\hat{\tilde{u}}_2\tilde{u}_{2x}\tilde{u}_{2y}I_\delta = \frac{\pi\tilde{s}_x\tilde{s}_y}{q^2\tilde{s}^3\tilde{u}_2}\left[3\left(\frac{1-q}{q}\tilde{s}+\hat{\tilde{\boldsymbol{s}}}\cdot\tilde{\boldsymbol{u}}\right)^2-\tilde{u}_2^2\right]H\left(\tilde{u}_2-|\frac{1-q}{q}\tilde{s}+\hat{\tilde{\boldsymbol{s}}}\cdot\tilde{\boldsymbol{u}}|\right).$$

One thus obtains

$$(II) = -\frac{8\epsilon K n\Theta}{3\pi^4}\left(\frac{3}{2\Theta}\right)^{1/2}\overline{\frac{\partial V_i}{\partial \tilde{r}_j}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\int_{|\frac{1-q}{q}\tilde{s}+\hat{\tilde{\boldsymbol{s}}}\cdot\tilde{\boldsymbol{u}}|}^{\infty}d\tilde{u}_2\frac{\tilde{u}_2}{\tilde{s}^3}\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2} \times\left[3\left(\frac{1-q}{q}\tilde{s}+\hat{\tilde{\boldsymbol{s}}}\cdot\tilde{\boldsymbol{u}}\right)^2-\tilde{u}_2^2\right]\hat{\Phi}_v(\tilde{u}_2)\hat{\Phi}_v(\tilde{u})\tilde{u}_x\tilde{u}_y\tilde{s}_x\tilde{s}_y. \tag{C 49}$$

Upon using the result: $\int_{\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{s}}=\tilde{u}\tilde{s}\cos\theta'}\mathrm{d}\hat{\tilde{\boldsymbol{u}}}\mathrm{d}\hat{\tilde{\boldsymbol{s}}}\tilde{u}_x\tilde{u}_y\tilde{s}_x\tilde{s}_y = \frac{4}{15}\pi^2\tilde{u}^2\tilde{s}^2(3\cos^2\theta'-1)$, followed by an appropriate shift in $\tilde{u}_2$, and substituting $\mathrm{d}y = \mathrm{d}(\cos\theta')$ one can perform the derivative with respect to ϵ at $\epsilon=0$ to obtain: $(II)=\zeta_3\epsilon n\ell\Theta^{1/2}\overline{\partial V_i/\partial r_j}$ where

$$\zeta_3=-\frac{8\sqrt{3}}{45\sqrt{2\pi}}\int_0^\infty \mathrm{d}\tilde{u}\int_0^\infty \mathrm{d}\tilde{u}_2\int_{-1}^{1}\mathrm{d}y\tilde{u}_2\tilde{u}^4(3y^2-1)(2\tilde{u}^2y^2-\tilde{u}_2^2)$$
$$\times\left\{-\tilde{u}y\left[\tilde{u}y+\frac{\pi^{1/2}}{2}(1+2\tilde{u}^2y^2)\mathrm{e}^{\tilde{u}^2y^2}(1+\mathrm{erf}(\tilde{u}y))\right]\right.$$
$$\times\left[1-\frac{\hat{\Phi}_v'\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)}{\hat{\Phi}_v\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)}-\frac{2}{2\tilde{u}^2y^2-\tilde{u}_2^2}\right]$$
$$\left.+1+\pi^{1/2}\tilde{u}y\,\mathrm{e}^{\tilde{u}^2y^2}(1+\mathrm{erf}(\tilde{u}y))\right\}\hat{\Phi}_v\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)\hat{\Phi}_v(\tilde{u})\mathrm{e}^{-\tilde{u}^2}\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}. \quad \text{(C 50)}$$

The integral in (C 50) has been evaluated numerically. The result is $\zeta_3\approx 0.1094$. The third part of $P_{ij}^{K\epsilon}$ is

$$P_{ij_3}^{K\epsilon}=-\frac{2n\Theta}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,\mathrm{e}^{-\tilde{u}^2}\tilde{\Omega}, \quad \text{(C 51)}$$

where $\tilde{\Omega}$ is given in (C 3). Substitution of (C 3) in (C 51) yields

$$P_{ij_3}^{K\epsilon}=-\frac{2n\Theta}{3\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}$$
$$\times(\Phi_K(\tilde{\boldsymbol{u}}_1')\Phi_\epsilon(\tilde{\boldsymbol{u}}_2')+\Phi_K(\tilde{\boldsymbol{u}}_2')\Phi_\epsilon(\tilde{\boldsymbol{u}}_1')-\Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_\epsilon(\tilde{\boldsymbol{u}}_2)-\Phi_K(\tilde{\boldsymbol{u}}_2)\Phi_\epsilon(\tilde{\boldsymbol{u}}_1)). \quad \text{(C 52)}$$

In (C 52) the relation between the primed and unprimed vectors is given by the elastic velocity transformation. Transforming the integral as in the derivation of (C 28) from (C 27), one obtains

$$P_{ij_3}^{K\epsilon}=-\frac{2n\Theta}{3\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\,\mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{d}\tilde{\boldsymbol{u}}\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}$$
$$\times(\Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_\epsilon(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_2)\Phi_\epsilon(\tilde{\boldsymbol{u}}_1))\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}I_\delta^{(0)}$$
$$+\frac{2n\Theta}{3\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\,\mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}$$
$$\times(\Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_\epsilon(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_2)\Phi_\epsilon(\tilde{\boldsymbol{u}}_1))\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}, \quad \text{(C 53)}$$

where $I_\delta^{(0)}$ is defined as in the above. Symmetry implies that only the viscous part of Φ_K contributes to the integrals of (C 53). The term $P_{ij_3}^{K\epsilon}$ is split into four parts. The first is given by (after employing the explicit forms of Φ_K and Φ_ϵ, and using $\tilde{\boldsymbol{s}}=\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1$)

$$(I)=-\frac{4\epsilon Kn\Theta}{3\pi^4}\left(\frac{3}{2\Theta}\right)^{1/2}\overline{\frac{\partial V_k}{\partial \tilde{r}_\ell}}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}$$
$$\times\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)(\tilde{u}_\ell-\tilde{s}_\ell). \quad \text{(C 54)}$$

Notice that except for the extra term $\hat{\Phi}_e(\tilde{u}_2)$ and the definition of $I_\delta^{(0)}$ for $q=1$, the integral in (C 54) is similar to the one in (C 45). Hence, performing a similar derivation to that following (C 45) one obtains $(I)=\zeta_4\epsilon n\ell\Theta^{1/2}\overline{\partial V_i/\partial r_j}$, where

$$\zeta_4=-\frac{128\sqrt{3}}{45\sqrt{2\pi}}\int_0^\infty \mathrm{d}\tilde{s}\int_0^\infty \mathrm{d}\tilde{u}\int_0^\infty \mathrm{d}\tilde{u}_2\int_{-1}^1 \mathrm{d}y\tilde{u}_2\tilde{s}\tilde{u}^4\left(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tfrac{1}{2}\tilde{s}^2(3y^2-1)\right)$$
$$\times\hat{\Phi}_v\left((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\right)\hat{\Phi}_e\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)\hat{\Phi}_v(\tilde{u})\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}. \quad \text{(C 55)}$$

The integral in (C 55) has been evaluated numerically. The result is $\zeta_4\approx 0.0015$.

The second term reads:

$$(II)=-\frac{4\epsilon Kn\Theta}{3\pi^4}\left(\frac{3}{2\Theta}\right)^{1/2}\overline{\frac{\partial V_k}{\partial \tilde{r}_\ell}}\int \mathrm{d}\tilde{\boldsymbol{s}}\,\mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\hat{\Phi}_v(\tilde{u}_2)\hat{\Phi}_e(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u})\overline{\overline{\tilde{u}_i\tilde{u}_j}}\tilde{u}_{2k}\tilde{u}_{2\ell}. \quad \text{(C 56)}$$

Except for the extra term $\hat{\Phi}_e(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)$ and the definition of $I_\delta^{(0)}$ for $q=1$, the integrand in (C 56) is similar to the one (C 48). Hence, following a similar derivation as in the above, it follows that $(II)=\zeta_5\epsilon n\ell\Theta^{1/2}\overline{\partial V_i/\partial r_j}$, where

$$\zeta_5=-\frac{32\sqrt{3}}{45\sqrt{2\pi}}\int_0^\infty \mathrm{d}\tilde{s}\int_0^\infty \mathrm{d}\tilde{u}\int_0^\infty \mathrm{d}\tilde{u}_2\int_{-1}^1 \mathrm{d}y\tilde{s}\tilde{u}_2\tilde{u}^4(3y^2-1)(2\tilde{u}^2y^2-\tilde{u}_2^2)$$
$$\times\hat{\Phi}_e\left((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\right)\hat{\Phi}_v\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)\hat{\Phi}_v(\tilde{u})\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}. \quad \text{(C 57)}$$

The integral in (C 57) has been evaluated numerically. The result is $\zeta_5\approx 0.0015$.

The third term reads (after performing the integral over $\hat{\boldsymbol{k}}$ and employing the explicit forms of Φ_K and Φ_ϵ)

$$(III)=\frac{4\sqrt{3}\epsilon n\ell\Theta^{1/2}}{3\sqrt{2\pi^3}}\overline{\frac{\partial V_k}{\partial r_\ell}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\hat{\Phi}_v^2(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_2)\overline{\overline{\tilde{u}_{1i}\tilde{u}_{1j}}}\tilde{u}_{1k}\tilde{u}_{1\ell}. \quad \text{(C 58)}$$

Tensorial arguments allow one to choose specific directions, e.g. $i=k=1$ and $j=\ell=2$, replace $\overline{\partial V_k/\partial r_\ell}$ by $\overline{\partial V_i/\partial r_j}$ and multiply the result by 2. One obtains

$$(III)=\frac{8\sqrt{3}\epsilon n\ell\Theta^{1/2}}{3\sqrt{2\pi^3}}\overline{\frac{\partial V_i}{\partial r_j}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\tilde{u}_{12}\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\hat{\Phi}_v^2(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_2)\tilde{u}_{1x}^2\tilde{u}_{1y}^2, \quad \text{(C 59)}$$

Next, performing the integration over all the orientations of $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$, keeping the angle between them fixed, one obtains: $\int_{\tilde{\boldsymbol{u}}_1\cdot\tilde{\boldsymbol{u}}_2=const}\mathrm{d}\hat{\tilde{\boldsymbol{u}}}_1\mathrm{d}\hat{\tilde{\boldsymbol{u}}}_2\tilde{u}_{1x}^2\tilde{u}_{1y}^2=\frac{8}{15}\pi^2\tilde{u}_1^4$. Using this result, it follows that $(III)=\zeta_6\epsilon n\ell\Theta^{1/2}\overline{\partial V_i/\partial r_j}$, where

$$\zeta_6=\frac{64\sqrt{3}}{45\sqrt{2\pi}}\int_0^\infty \mathrm{d}\tilde{u}_1\int_0^\infty \mathrm{d}\tilde{u}_2\tilde{u}_1^6\tilde{u}_2^2R_0(\tilde{u}_1,\tilde{u}_2)\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\hat{\Phi}_v^2(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_2). \quad \text{(C 60)}$$

The integral in (C 60) has been evaluated numerically. The result is $\zeta_6\approx 0.0010$.

The fourth term reads

$$(IV)=\frac{4\sqrt{3}\epsilon n\ell\Theta^{1/2}}{3\sqrt{2\pi^3}}\overline{\frac{\partial V_k}{\partial r_\ell}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\overline{\overline{\tilde{u}_{1i}\tilde{u}_{1j}}}\tilde{u}_{2k}\tilde{u}_{2\ell}, \quad \text{(C 61)}$$

and it equals (following similar tensorial considerations)

$$(IV)=\frac{8\sqrt{3}\epsilon n\ell\Theta^{1/2}}{3\sqrt{2\pi^3}}\overline{\frac{\partial V_i}{\partial r_j}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{1x}\tilde{u}_{1y}\tilde{u}_{2x}\tilde{u}_{2y}. \quad \text{(C 62)}$$

Performing the integral over all orientations of $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$ while keeping the angle between them, θ', fixed, yields $\int_{\tilde{\boldsymbol{u}}_1\cdot\tilde{\boldsymbol{u}}_2=\tilde{u}_1\tilde{u}_2\cos\theta'}\mathrm{d}\hat{\tilde{\boldsymbol{u}}}_1\mathrm{d}\hat{\tilde{\boldsymbol{u}}}_2\tilde{u}_{1x}\tilde{u}_{1y}\tilde{u}_{2x}\tilde{u}_{2y}=\frac{4\pi^2\tilde{u}_1^2\tilde{u}_2^2}{15}(3\cos^2\theta'-1)$.

It follows that $(IV) = \zeta_7 \epsilon n \ell \Theta^{1/2} \overline{\partial V_i / \partial r_j}$ where

$$\zeta_7 = \frac{64\sqrt{3}}{45\sqrt{2}\pi} \int_0^\infty \mathrm{d}\tilde{u}_1 \int_0^\infty \mathrm{d}\tilde{u}_2 \tilde{u}_1^4 \tilde{u}_2^4 R_2(\tilde{u}_1, \tilde{u}_2) \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \hat{\Phi}_v(\tilde{u}_1) \hat{\Phi}_v(\tilde{u}_2) \hat{\Phi}_e(\tilde{u}_1), \qquad \text{(C 63)}$$

and R_2 is given by (for $\tilde{u}_1 > \tilde{u}_2$) $R_2(\tilde{u}_1, \tilde{u}_2) = 2\tilde{u}_2^4/35\tilde{u}_1^3 - 2\tilde{u}_2^2/15\tilde{u}_1$. The integral has been evaluated numerically. The result is: $\zeta_7 \approx -0.0003$. Summing all contributions to $P_{ij}^{K\epsilon}$ one obtains that

$$P_{ij}^{K\epsilon} = -2\epsilon \mu_1 n \ell \Theta^{1/2} \overline{\frac{\partial V_i}{\partial r_j}}, \qquad \text{(C 64)}$$

where $\mu_1 \approx 0.0576$.

Appendix D is available on request from the authors or the JFM Editorial Office.

Appendix E is available on request from the authors or the JFM Editorial Office.

REFERENCES

Bobylev, V. A. 1984 Exact solutions of the nonlinear Boltzmann equation and the theory of Maxwell gas relaxation. *Theor. Math. Phys.* **60**, 280–310.

Boyle, E. J. & Massoudi, M. 1990 A theory for granular materials exhibiting normal stress effects based on Enskog's dense gas theory. *Intl J. Engng Sci.* **28**, 1261–1275.

Campbell, C. S. 1990 Rapid granular flows. *Ann. Rev. Fluid Mech.* **22**, 57–92.

Cercignani, C. 1975 *Theory and Application of the Boltzmann Equation* Scottish Acad. Press, Edinburgh and London.

Chapman, S. & Cowling, T. G. 1970 *The Mathematical Theory of Nonuniform Gases.* Cambridge University Press.

Fiscko, K. A. & Chapman, D. R. 1989 Comparison of Burnett, super-Burnett and Monte-Carlo solutions for hypersonic shock structure. *Prog. Aeronaut. Astronaut.* **118**, 374–395.

Goldhirsch, I. & Sela, N. 1996 Origin of normal stress differences in rapid granular flows. *Phys. Rev.* E **54**, 4458–4461.

Goldshtein, A. & Shapiro, M. 1995 Mechanics of collisional motion of granular materials, Part I: General hydrodynamic equations. *J. Fluid Mech.* **282**, 75–114.

Grad, H. 1949 On the kinetic theory of rarefied gases. *Commun. Pure Appl. Maths* **2**, 331–407.

Haff, P. K. 1986 Grain flow as a fluid mechanical phenomenon. *J. Fluid Mech.* **134**, 401–948.

Harris, S. 1971 *Introduction to the Theory of the Boltzmann Equation.* Holt, Reinhart and Winston.

Herrmann, H. J. 1992 Simulation of granular media. *Physica* A **191**, 263–276.

Hutter, K. & Rajagopal, K. R. 1994 On flows of granular materials. *Continuum Mech. Thermodyn.* **6**, 81–139.

Jaeger, H. M., Nagel, S. R. & Behringer, R. P. 1996 The physics of granular materials. *Physics Today* April 1996, 32–38.

Jenkins, J. T. & Richman, M. W. 1985*a* Grand's 13-moment system for a dense gas of inelastic particles. *Arch. Rat. Mech. Anal.* **87**, 355–377.

Jenkins, J. T. & Richman, M. W. 1985*b* Kinetic theory for plane flows of a dense gas of identical, rough, inelastic circular disks. *Phys. Fluids* **28**, 3485–3494.

Jenkins, J. T. & Richman, M. W. 1988 Plane simple shear of smooth inelastic circular disks: the anisotropy of the second moment in the dilute and dense limit. *J. Fluid. Mech.* **192**, 313–328.

Kogan, M. K. 1969 *Rarefied Gas Dynamics.* Plenum.

Lun, C. K. K., Savage, S. B., Jeffrey, D. J. & Chepurniy, N. 1984 Kinetic Theories of granular flow: inelastic particles in a Couette flow and slightly inelastic particles in a general flow field. *J. Fluid Mech.* **140**, 223–256.

Lun, C. K. K. 1991 Kinetic theory for granular flow of dense, slightly inelastic, slightly rough spheres. *J. Fluid Mech.* **223**, 539–559.

Pekeris, C. L. 1955 Solution of the Boltzmann-Hilbert integral equation. *Proc. Natl Acad. Sci.* **41**, 661–664.

Rosenau, P. 1989 Extending hydrodynamics via the regularization of the Chapman–Enskog expansion. *Phys. Rev.* A **40**, 7193–7196.

Sela, N., Goldhirsch, I. & Noskowicz, S. H. 1996 Kinetic Theoretical Study of a simply sheared two dimensional granular gas to Burnett order. *Phys. Fluids* **8**, 2337–2353.

Walton, O. R. & BRAUN, R. L. 1986 Stress calculations for assemblies of inelastic spheres in uniform shear. *Acta Mech.* **63**, 73–86.